

Math 5660 Fall 2002 Homework 2 solution

Let p be a positive number. What is the value of $x = \sqrt{p + \sqrt{p + \dots}}$. Note that this can be interpreted as meaning $x = \lim_{n \rightarrow \infty} x_n$, where $x_1 = \sqrt{p}$, $x_2 = \sqrt{p + \sqrt{p}}$, and so on. Hint: Observe that $x_{n+1} = \sqrt{p + x_n}$.

Solution. The desired value x is the limit of functional iterations $x_{n+1} = f(x_n)$, where $x_0 = 0$ and $f(t) = \sqrt{p + t}$. First we need to show that the iterations converge. Compute the derivative, $f'(t) = \frac{1}{2\sqrt{p+t}}$. For small t , the value of the derivative can be larger than one, so the Banach contraction principle cannot be applied immediately. Few quick experiments show that the iterates are increasing, so we prove that by induction. We have $x_1 = \sqrt{p} > x_0 = 0$. Assume that $x_n > x_{n-1}$. Because $f' > 0$, f is increasing on $[0, +\infty)$, and it follows that $x_{n+1} = f(x_n) > f(x_{n-1}) = x_n$. So the sequence $\{x_n\}$ is indeed increasing. To show that it has a limit, all we need to prove is that it is bounded. This follows from the fact that $\lim_{t \rightarrow +\infty} \sqrt{p+t} - t = -\infty < 0$, so, for sufficiently large t , $f(t) < t$. Hence, the sequence x_n must be bounded, otherwise it would not be increasing for sufficiently large n . We conclude that the sequence x_n is increasing and bounded, hence it has a limit.

Now $\lim_{n \rightarrow \infty} x_n = x$ and f is continuous, hence $x = f(x)$. This gives $x = \sqrt{p + x}$, $x > 0$. Solving the quadratic equation $x^2 = p + x$ we get $x = \frac{p \pm \sqrt{1+4p}}{2}$, and, because we know that $x > 0$, the positive sign applies, giving $x = \frac{p + \sqrt{1+4p}}{2}$.

Note: The essential points of the solution are:

1. Show that the sequence x_n converges
2. Apply continuity of f to get $x = f(x)$
3. Use the fact that $x > 0$ to choose the correct root of the quadratic equation