ARC-TRACEABLE TOURNAMENTS

by

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Arc-traceable Tournaments

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ABSTRACT

A tournament T is arc-traceable when each arc of the tournament is a part of a hamiltonian path. We characterize arc-traceable upset tournaments and show that this property is independent of the number of hamiltonian paths in such tournaments. We show that non-arc-traceable tournaments have a specific structure, and give several sufficient conditions for strong tournaments to be arc-traceable, including Dirac-like minimum degree conditions, Ore-like conditions, and irregularity conditions. In each case, we show that the result is best possible. Next, we give some extremal results for arc-traceable tournaments, bounding the number of arcs in a tournament that are not part of any hamiltonian path, and finding the minimum number of vertices in a k-arc-strong tournament that is not arc-traceable. Finally, we also consider the arc-traceable property in two other classes of digraphs, semicomplete digraphs and local tournaments, both of which share many properties with tournaments.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________________
Michael S. Jacobson

DEDICATION

I would like to dedicate this dissertation to my parents. Their encouragement and support has been unwavering, and their patience almost unlimited.

ACKNOWLEDGMENT

First and foremost, I would like to thank my advisor, Mike Jacobson. I would like to think that my frequent interruptions of his work were usually a welcome distraction, but they were certainly a distraction. I am grateful for the time he has spent with me on this thesis and on other projects that have collectively made me a much better mathematician. In addition, I want to thank both Rich Lundgren and to Bill Cherowitzo, who encouraged and mentored me early in my graduate studies and knew long before I did that my education would culminate in a Ph.D. Bill, in particular, deserves credit for trying his best to turn me into a geometer. Ultimately unsuccessful, I learned a great deal in the process.

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1. Introduction and background

1.1 Prologue

Questions relating to paths and cycles in graphs and digraphs have been studied extensively from a variety of perspectives. One of the most popular questions in undirected graphs relates to determining the class of graphs that contain a hamiltonian path or cycle. This basic question has been generalized and specialized in many ways which has led to a number of deep and interesting results (for a survey of such results, see [15]).

Our particular interest is a property of digraphs we call *arc-traceable*; an arc-traceable digraph is one in which every arc is part of a path that includes every vertex, i.e., a hamiltonian path. This question seems more difficult than the related question for undirected graphs, studied by Balińska, Zwierzyński, Gargano and Quintas ([2], [3] and [4]). As a result, we investigate arc-traceability in the sub-class of digraphs known as *tournaments*. Tournaments have additional structure that, in many cases, can be shown to guarantee arc-traceability. We use this additional structure to investigate tournaments that are not arc-traceable.

In the next section, we provide some brief background material with an emphasis on terminology and classical results related to digraphs and tournaments. In Chapter 2, we review some more recent results specifically relating to paths and cycles in digraphs and tournaments, and we develop a necessary condition for a digraph to be arc-traceable. This necessary condition is used to show that the study of arc-traceable tournaments can be restricted to studying arc-traceable strong tournaments.

Initially, we consider the question in the context of a sub-class of strong tournaments known as upset tournaments, and in Chapter 3, we give a characterization of arc-traceable upset tournaments. We use this characterization to produce examples of tournaments that have many arcs that are not part of any hamiltonian path (such arcs will be called *non-traceable*). We also develop some techniques to demonstrate that arc-traceability is independent of the number of hamiltonian paths in upset tournaments. In other words, arc-traceability in a tournament is not merely a consequence of having many hamiltonian paths.

In Chapter 4, we broaden the investigation of arc-traceability to all strong tournaments. We begin with some simple observations about sufficient conditions for a strong tournament to be arc-traceable. From these basic observations, we then develop some necessary conditions for a strong tournament to be non-arc-traceable. These conditions, in turn, yield many new and improved sufficient conditions for arc-traceable tournaments, and a structure that we use to construct minimal non-arc-traceable tournaments with many arc-disjoint paths between every pair of vertices. We also use the structure of non-arc-traceable tournaments to determine the maximal number of non-traceable arcs in a strong n-tournament.

In Chapter 5, we briefly consider arc-traceability for digraphs in general. We investigate arc-traceability in two classes of digraphs that include tournametns, *semicomplete* digraphs and *local tournaments*. We also briefly consider some questions closely related to arc-traceability in tournaments and these superclasses of tournaments. The structure developed in Chapter 4 features prominently in these questions, and we give examples that show how related structures occur in these contexts. These related questions suggest several directions for future research.

1.2 Background

We begin with some of the fundamental results related to directed graphs in general and tournaments in particular. We assume the reader is familiar with the study of undirected graphs and suggest [10] as both a general resource and for more detailed background on the following results for digraphs and [33] as a resource for more information on tournaments.

A digraph $D = (V, A)$ consists of a set V called *vertices* and a collection A of ordered pairs of elements from V called *arcs*. Just as in the case of undirected graphs the *order* of D is the number of vertices in V and the *size* of a digraph is the number of arcs in A. Typically, the study of digraphs is restricted to those digraphs in which $vv \notin A$ for every $v \in V$ and there is at most one copy of each of the remaining ordered pairs. Such a digraph is called *strict*, and when D is a strict digraph, the collection of arcs is more commonly referred to as a set. Strict digraphs are in a sense analogous to *simple* undirected graphs; graphs with no loops and at most one edge joining each pair of vertices. Unless specifically stated otherwise, we will assume that all digraphs are strict. For convenience, we often simply write uv to represent the arc (u, v), and we will say that u *dominates* v and write $u \to v$ when (u, v) is an arc of D . A *subdigraph* of a digraph $D = (V, A)$ is a digraph $D' = (S, T)$ where both $S \subseteq V$ and $T \subseteq A$. For a set $S \subset V$, we define the *induced subdigraph* $D[S]$ as the digraph with vertex set S and arc set $T = \{(u, v) \mid (u, v) \in A \text{ and } u, v \in S\}$. In other words, the arcs of an induced subdigraph include all arcs that join two vertices in S. In

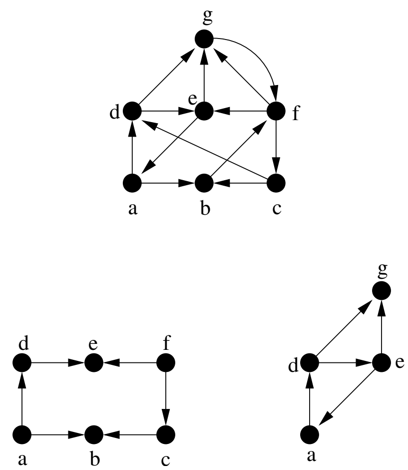


Figure 1.1: A digraph and two subdigraphs, one induced.

Figure 1.1, we show an example of a strict digraph along with two subdigraphs. The subdigraph on the right is induced, while the subdigraph on the left is not; it includes the vertices c and d but not the arc from c to d.

The out-neighborhood of a vertex v is the set $\{u \mid vu \in A\}$ and is denoted $N_D^+(v)$ or $N^+(v)$ when the context is clear. Similarly, $N^-(v) = \{u \mid uv \in A\}$. The out-degree ($d^+(v)$) and in-degree ($d^-(v)$) are defined as $|N^+(v)|$ and $|N^-(v)|$, respectively. Similar to graphs, we then let δ^+ (δ^-) be the smallest out-degree (in-degree) and Δ^+ (Δ^-) be the largest out-degree (in-degree) among all the vertices of D. Additionally, we set $\delta^0 = \min(\delta^+, \delta^-)$ and $\Delta^0 = \max(\Delta^+, \Delta^-)$. When we are working with more than one digraph, we will use the notation $\delta^0(D)$, for example, to clarify the digraph to which we are referring.

A *path* in a digraph is a sequence of distinct vertices $v_1v_2 \dots v_k$ such that $v_iv_{i+1} \in A$ for each $i, 1 \leq i < k$. If, in addition, $v_kv_1 \in A$, then the sequence is a *cycle*. The *order* of a path or cycle is the number of vertices in the sequence, and the *length* is the number of arcs. Thus, a path of length l has order $l-1$, while a cycle of length l also has order l. Note, just as for graphs, any consecutive subsequence of the vertices of a path or cycle is also a path. Conversely, we will frequently want to combine two or more paths into larger paths. If $P = v_1v_2\dots v_k$ and $Q = u_1u_2\dots u_m$ are vertex disjoint paths, and $v_ku_1 \in A$, then we will denote the combined path $v_1\dots v_ku_1\dots u_m$ simply as PQ. A digraph D which contains no cycles is called *acyclic*.

Of particular interest is a path or cycle containing all the vertices of D, i.e. a *hamiltonian* path or cycle. Just as for graphs, a digraph containing a hamiltonian cycle will be called hamiltonian and a digraph containing a hamiltonian path will be called *traceable* (of course, every hamiltonian digraph is traceable but not conversely). Figure 1.2 shows a digraph that is hamiltonian, with a hamiltonian cycle highlighted by dashed arcs.

For undirected graphs, hamiltonian paths and cycles have been studied extensively. Two classic, early results are due to Dirac and Ore.

Theorem A (Dirac [11]). *If G is a simple graph with at least three vertices and $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian.*

Theorem B (Ore [30]). *Let G be a simple graph. If $d(u)+d(v) \geq n$ for every pair of distinct non-adjacent vertices u and v, then G is hamiltonian.*

We note that for strict digraphs, an analogue of Theorem A was given by Ghouilá-Houri.

Theorem C (Ghoilá-Houri [13]). *If D is a strict digraph with $\delta^0 \geq \frac{n}{2}$, then D is hamiltonian.*

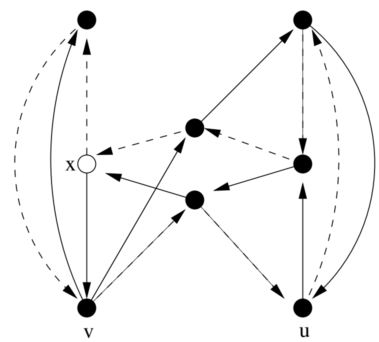


Figure 1.2: A hamiltonian digraph

Similarly, Theorem B has been generalized to strict digraphs by Woodall [43]. Meyniel improved both of these results with the following:

Theorem D (Meyniel [26]). *If D is a strongly connected strict digraph and $d^+(u) + d^-(u) + d^+(v) + d^-(v) \geq 2n - 1$ for every pair of vertices u and v such that neither $uv \in A(D)$ nor $vu \in A(D)$, then D is hamiltonian.*

A digraph is *strongly connected* or simply *strong* when for every pair of distinct vertices u and v there exists both a path from u to v and a path from v to u. Note, a single vertex digraph is always strong. As such, for any digraph D, each vertex is contained in a strongly connected induced subdigraph of D. When such a strong induced subdigraph is maximal, it is called a *strong component* of D, and it is easy to see that these strong components must, in fact, partition V. We then define a new digraph D^*, called the *condensation* of D by letting $V(D^*)$ be the set of strong components of D and $A(D^*) = \{D_iD_j \mid uv \in A$ for some $u \in D_i, v \in D_j$ with $i \neq j\}$. Again, it is easy to see that D^* is an acyclic digraph. For more information about the condensation of a digraph, see [36].

In a strong digraph D, if the induced subdigraph $D[S]$ is not strongly connected for some $S \subset V$, then $V \setminus S$ is called a *cut-set* of D. When $D[S]$ is strong for all sets S with $|S| \geq |V| - (d-1)$, we say that D is *d-connected* and we define $\kappa(D)$ to be the smallest d such that D is d-connected. Analogously, if $D' = (V, T)$ is not strongly connected for some $T \subset A$, then $A \setminus T$ is called an *arc cut-set*, and we define $\kappa'(D)$ as the smallest k such that $D' = (V, T)$ is not strong for some $T \subset A$ with $|T| = |A| - k$. We then we say that D is *k-arc-connected* or *k-arc-strong* whenever $k \leq \kappa'(D)$. The digraph in Figure 1.2 is 2-arc-strong; both the solid arcs and the dashed arcs form hamiltonian cycles, so removing any single arc still leaves a path between any two vertices. Note, this digraph is not 2-connected, however, since removing the vertex x leaves no path from u to v.

When two distinct non-adjacent vertices x and y are specified, an x,y-separating-set is a set S of vertices such that every path from x to y contains at least one vertex of S. Similarly, an x,y-separating-set of arcs is a set T of arcs such that every path from x to y contains at least one arc of T (in this case, we need not require that x and y be non-adjacent). In both cases, it is easy to see that when there exist m paths from x to y that are internally vertex disjoint (arc-disjoint) then the set S (T) must contain at least one vertex (arc) from each path. The converse also holds in both cases, a result referred to as Menger's Theorem. Menger's Theorem is stated in a wide variety of ways, both for graphs and digraphs. We give two variations here.

Theorem E (Menger [25]). *Let D be a digraph and let x and y be distinct non-adjacent vertices of D. The number of internally vertex disjoint paths from*

x to y equals the size of the smallest x, y-separating set. Similarly, for any distinct (possibly adjacent) vertices x and y, the number of arc-disjoint paths from x to y equals the size of the smallest x, y-separating set of arcs.

Our main focus will be on a special class of digraphs called *tournaments*. A tournament is a directed graph D such that for any two distinct vertices u and v, either $uv \in A(D)$ or $vu \in A(D)$ but not both. Typically, a tournament is denoted by T rather than D, and we will say T is an n-tournament when $|V(T)| = n$. It is often helpful to think of an n-tournament as an orientation of the complete graph K_n. From this perspective, it is immediate that a tournament has no self-loops or cycles of length two. It is also clear that $|A(T)| = |E(K_n)| = \binom{n}{2}$ and that $N^-(v)$ and $N^+(v)$ partition the set $V(T) \setminus \{v\}$ regardless of our choice of v. This also shows that $d^+(v) + d^-(v) = n - 1$ for every vertex v. In accordance with the name tournament, we will often refer to $d^+(v)$ as the *score* of the vertex v. We illustrate an 8-tournament in Figure 1.3. Clearly, drawing even relatively small tournaments is impractical due to the large number of arcs.

It is also apparent, by considering the tournament as an orientation of a complete graph, that any induced subdigraph of an n-tournament is also a tournament. We simply refer to such an induced subdigraph as a *subtournament*, and for the set $U \subset V(T)$ we denote the subtournament on the set U by $T[U]$. When $U = V(T) \setminus S$ for some subset S, we simply write $T - S$ instead of $T[V \setminus S]$. If $S = \{v\}$, then we will abbreviate notation further and refer to $T - \{v\}$ as T_v.

For any two subsets A and B of $V(T)$, there are exactly $|A||B|$ arcs between these two sets. When uv is an arc of T for every $u \in A$ and $v \in B$, we will

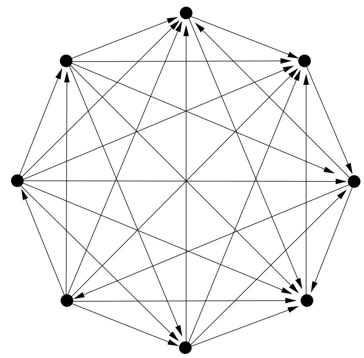


Figure 1.3: A tournament with eight vertices.

say A *dominates* B. Note, just as the tournament as a whole can be viewed as an orientation of the complete graph, the arcs between the sets A and B can be viewed as an orientation of the complete bipartite graph $K_{|A|,|B|}$. When A dominates B then this orientation is such that the arcs between A and B always originate in A and terminate in B.

In addition to the traditional measures of in- and out-degrees in a tournament such as δ^+, δ^- and δ^0, we also measure the *irregularity* of a tournament, denoted $i(T)$ and defined as $\max_{v \in V} |d^+(v) - d^-(v)|$. From this definition, we see that $i(T) = 0$ if and only if every vertex has equal in- and out-degree and all of these values must then be exactly $\frac{n-1}{2}$ (and hence, n must be odd). Such a tournament is called *regular*. When n is even, $i(T) \geq 1$ and in the case of equality, the vertices have out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$. Such tournaments are called *near-regular* or *almost-regular*.

Proposition 1.1. $i(T) = n - 1 - 2\delta^0$

Proof. If $|d^+(v) - d^-(v)| > n - 1 - 2\delta^0$ for some $v \in V$, then we must have either

$d^+(v) - d^-(v) > n - 1 - 2\delta^0$ or $d^-(v) - d^+(v) > n - 1 - 2\delta^0$. Substituting in $d^+(v) + d^-(v)$ for $n-1$ into both equations and simplifying, we obtain either $d^-(v) < \delta^0$ or $d^+(v) < \delta^0$, a contradiction. Thus, $i(T) \leq n - 1 - 2\delta^0$. To complete the proof, note that

$$\begin{aligned}
\max_{v \in V} |d^+(v) - d^-(v)| &\geq \max(\Delta^+ - \delta^-, \delta^+ - \Delta^-) \\
&\geq \max\Big((n - 1 - \delta^-) - \delta^-, \delta^+ - (n - 1 - \delta^+)\Big) \\
&\geq \max(n - 1 - 2\delta^-, n - 1 - 2\delta^+) \\
&\geq n - 1 - 2\min(\delta^-, \delta^+) \\
&\geq n - 1 - 2\delta^0.
\end{aligned}$$

□

For any tournament $T = (V, A)$, we can order the vertices $v_1, v_2, \ldots, v_n$ in such a way that $d^+(v_i) \leq d^+(v_j)$ for every $i < j$. If we let $d^+(v_i) = s_i$ for $1 \leq i \leq n$, this ordering gives a non-decreasing sequence $S = s_1, s_2, s_3, \ldots, s_n$ of length n. This sequence is called the *score sequence* of the tournament T. It is clear that $\sum_{i=1}^{n} s_i = |A| = \binom{n}{2}$ since each arc is counted exactly once in the sequence. Also, the subset $V_k = \{v_1, v_2, \ldots, v_k\}$ induces a subtournament for each $k < n$, so we also must have $\sum_{i=1}^{k} s_i \geq |A(T[V_k])| = \binom{k}{2}$ for each $k < n$. Landau [23] showed that these necessary conditions are also sufficient. Thus, any non-decreasing sequence $S = s_1, s_2, \ldots, s_n$ such that

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \text{ for each } k, 1 \leq k < n$$

and

$$\sum_{i=1}^{n} s_i = \binom{n}{2}$$

is the score sequence for some tournament T.

A tournament is *transitive* if $A(T)$ induces a transitive (and hence acyclic) relation on the set $V(T)$. Furthermore, since $A(T)$ is anti-symmetric and every pair of vertices is joined by an arc, every pair of vertices is comparable under this transitive relation and thus gives a total ordering of the set $V(T)$. If $V(T) = \{v_1, \ldots, v_n\}$, with the subscripts corresponding to this order, then $v_i v_j \in A(T)$ if and only if $i > j$ and hence $d^+(v_i) = i-1$ (or, of course, $v_i v_j \in A(T)$ if and only if $i < j$, in which case $d^-(v_i) = i-1$; we generally prefer the former ordering). It is well known (c.f. [36]) that a tournament is transitive if and only if it is acyclic.

Note, if a tournament T is strongly connected, the condensation digraph T^* consists of a single vertex, which is trivially transitive. If T is not strongly connected, then because T is an orientation of the complete graph, the underlying graph of T^* is also complete, and so T^* is also a tournament. Furthermore, T^* is by definition acyclic, and hence T^* is a transitive tournament.

If $T = (V, A)$, for any $B \subseteq A$, we can form a new tournament T' by "reversing" the arcs of B. More specifically, let $\overline{B} = \{uv \mid vu \in B\}$, then $T' = (V, (A \setminus B) \cup \overline{B})$. Two special cases are of particular interest. First, when $B = A$, we will call T' the *reversal* of T and denote this tournament $\overline{T}$. Note, paths and cycles in T are also paths and cycles in $\overline{T}$. The second special case is when the T' obtained is a transitive tournament, and in this case we can use the transitive ordering of $V(T')$ on $V(T)$ as well. Using this ordering, in which

v_iv_j is an arc of T' whenever $i > j$, B is precisely the set or arcs v_iv_j in T such that $i < j$ and the arcs of B are sometimes referred to as the *upset arcs* of the tournament. Using this idea, we develop a simplified way to draw a tournament. We place the vertices in a linear order, and make the arc from u to v implicit if u comes after v in this ordering. In other words, we only draw the arcs of B. Figure 1.4 uses this technique to represent the same tournament shown in Figure 1.3. In all subsequent figures, we will use this technique to drawn tournaments as well as similar digraphs. In some cases, we may also draw a small number of the non-upset arcs when these arcs are of particular interest (to highlight non-traceable arcs, for example). Note, each ordering of $V(T)$ determines a set of upset arcs B and if we choose the ordering badly, the set B can be large, which minimizes the advantage of drawing only the upset arcs. We will avoid such orderings. Further simplifying our figures, we will use a similar technique for the condensation of tournaments that are not strong; we will represent the non-trivial components of the tournament with large circles or ellipses, often without specifying the internal structure of these strong components. Furthermore, using a top-to-bottom or left-to-right ordering of these components, we need not draw in any arcs at all as the condensation is a transitive tournament. Finally, we will sometimes wish to illustrate most of the arcs of a tournament, but leave the direction of one or more arcs unspecified. We will indicate such unspecified arcs with a dashed line.

When $V(T)$ has some ordering such that the set $B = \{v_iv_j \mid i < j\}$ is the arc set of a path U from v_1 to v_n, T is called an *upset tournament* and the path U is called the *upset path* of T. Since reversing the arcs of any path

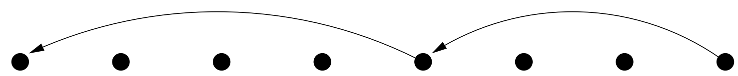


Figure 1.4: A simplified representation of a tournament

leaves the degrees of the interior vertices unchanged, $d^+_T(v_i) = d^+_{T'}(v_i) = i - 1$ for each $i, 2 \leq i \leq n-1$. Similarly, $d^+_T(v_1) = d^+_{T'}(v_1) + 1 = 0 + 1 = 1$ and $d^+_T(v_n) = d^+_{T'}(v_n) - 1 = n - 1 - 1 = n - 2$. Thus, an upset tournament has score sequence $1, 1, 2, \ldots, n-3, n-2, n-2$. Brualdi and Li [9] showed the converse of this result is also true; that any tournament with this sore sequence is an upset tournament. The tournament in Figures 1.3 and 1.4 is an example of an upset tournament. These two figures also show the advantage of using our simplified representation in the case of upset tournaments; it is easy to see that the tournament in Figure 1.4 is an upset tournament, but this is far from clear by looking at the same tournament in Figure 1.3 with all the arcs drawn.

Upset tournaments form a very natural sub-class of tournaments, because they are very nearly hereditary. While it is not true that every subtournament of an upset tournament is itself an upset tournament, it turns out that every non-trivial strong component of such a subtournament is an upset tournament. In particular, every strong subtournament of an upset tournament is an upset tournament.

2. Paths and cycles in tournaments

The study of hamiltonian paths and cycles in tournaments began with the very early result of Rédei.

Theorem F (Rédei [32]). *Every tournament is traceable.*

Similarly, strong tournaments were shown to be hamiltonian independently by Foulks and by Harary and Moser (this is also implied by Theorem D).

Theorem G (Foulkes [12] and Harary and Moser [18]). *Every strong tournament is hamiltonian.*

As a result, the investigation of hamiltonian paths and cycles in tournaments focuses on stronger conditions than the existence of at least one hamiltonian path or cycle. We survey a few of these stronger conditions, beginning with results for cycles in tournaments, and then turning our attention to paths.

2.1 Cycles in tournaments

One of the first results generalizing hamiltonicity in tournaments is due to Moon. He showed that every strongly connected tournament has a property he called *vertex-pancyclic.*

Theorem H (Moon [29]). *If T is a strong tournament, and v is any vertex of T, then v is in a cycle of length k for $3 \leq k \leq n$.*

We note that a similar property does not hold for the arcs of a strong tournament. Consider the tournament with six vertices shown in Figure 2.1. Reversing the arc a from the bottom vertex to the top vertex gives a tournament T' with two strong components, each of order three. Any cycle of this tournament that does not use the arc a is also a cycle of T' and hence has order at most three. Since no cycle can contain both a and either of the other upset arcs shown, these upset arcs can not be on any cycle of length four or more.

A natural question arises; what conditions on a tournament guarantee that every arc is on a cycle of all possible lengths? We will call tournaments in which every arc is contained in a cycle of length exactly k *arc-k-cyclic*, and if a tournament is arc-k-cyclic for each possible k, (in other words, for $3 \leq k \leq n$), then the tournament is called *arc-pancyclic*. In the special case where $k = n$, we will say a tournament is *arc-hamiltonian* rather than arc-n-cyclic. Alspach [1] initially showed that every regular tournament is arc-pancyclic. Jakobsen [21] studied near-regular tournaments and found that while not necessarily arc-pancyclic, for $n \geq 8$, such tournaments are arc-k-cyclic for each $k, 4 \leq k \leq n$. In 1980, Thomassen [38] showed that every 3-connected tournament is arc-pancyclic and that there are infinitely many 2-connected tournaments that are not arc-hamiltonian (and hence, not arc-pancyclic). Thomassen was also able to give a sufficient condition for almost arc-pancyclic tournaments based on irregularity.

Theorem I. *Thomassen [38] If T is an n-tournament with $i(T) \leq \frac{n-3}{5}$, then T is k-arc-cyclic for $4 \leq k \leq n$*

As regular and near-regular tournament have irregularity zero and one, respectively, this result implies (along with the simple observation that for regular tournaments, each arc is in a cycle of length three) the results of both Alspach

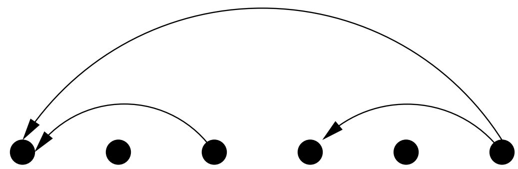


Figure 2.1: A strong tournament that is not arc-pancyclic

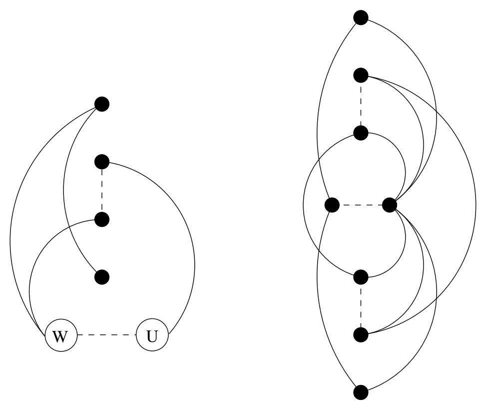


Figure 2.2: The two families of arc-3-cyclic tournaments that are not arc-pancyclic.

and Jakobsen.

For $k = 3$, Moon [27] was able to show that nearly all tournaments are arc-3-cyclic. This last result was then used by Tian, Wu and Zhang to aid in characterizing arc-pancyclic tournaments. They showed the following:

Theorem J (Tian, Wu and Zhang [41]). *An arc-3-cyclic tournament is arc-pancyclic unless it belongs to one of the families D_6 or D_8 shown in Figure 2.2.*

From this result, one can deduce the above results of Thomassen, and hence also of Jakobsen and Alspach.

2.2 Paths in tournaments

Properties related to similar generalizations of traceability in tournaments are not as well studied. The work of Rédei [32] showed that not only does every tournament have a hamiltonian path, but that every tournament has an odd number of such paths. Further, it is a simple exercise (cf. [36]) to demonstrate that a tournament has a unique hamiltonian path if and only if it is acyclic (or equivalently, is transitive). The number of distinct hamiltonian paths in a tournament has been studied, and it has been shown by Moon [28] that for strong tournaments, the number of such paths grows exponentially with n. For non-strong tournaments, the number of distinct hamiltonian paths is simply the product of the number of hamiltonian paths in each strong component.

The question of which arcs in a digraph can be extended to a hamiltonian path was first suggested by Balińska, Gargano and Quintas [2] in a paper analyzing the analogous concept in undirected graphs. We use the term *arc-traceable* to describe digraphs with this property. In this undirected context, it was shown that every hamiltonian graph is edge-traceable (called *totally-traceable* in [2], [3] and [4]). Since hamiltonian graphs are extensively studied, this gives a great deal of information about edge-traceable graphs. In the directed case, examples abound which show that hamiltonicity is not sufficient to guarantee arc-traceability. One such example is shown in Figure 2.3.

The lack of a well known sufficient condition for a digraph to be arc-traceable would seem to make the problem significantly more difficult than in the undirected case. Apart from some very special cases mentioned in [2], we note the following necessary condition.

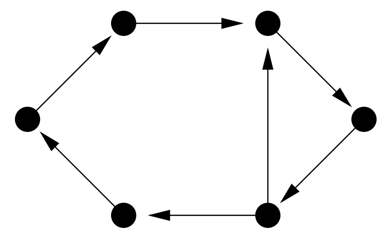


Figure 2.3: A hamiltonian digraph that is not arc-traceable.

Theorem 2.1. *If D is an arc-traceable digraph, then D^*, the condensation of D is a path.*

Proof. Let D be an arc-traceable digraph. Assume D has m strong components and let D^* be the condensation of D. If D is traceable, D^* must also be traceable. So, it suffices to show that when D is arc-traceable, then D^* is an oriented tree, as the only traceable oriented tree is a path. We establish this by showing that the underlying graph of D^* is acyclic.

Since D^* is acyclic, it contains some vertex of in-degree 0. Label such a vertex v_1^*. Now $D^* - \{v_1^*\}$ is also acyclic, and so this subgraph also has at least one vertex of in-degree 0. Choose such a vertex and label it v_2^*. Proceeding inductively, we can label all the vertices of D^* as $v_1^*, v_2^*, \ldots, v_m^*$ such that v_p has in-degree 0 in $D^* - \{v_1, \ldots, v_{p-1}\}$. Thus, for any $p > q$, $v_p^* v_q^* \notin A(D^*)$. This well known ordering is known as a topological sorting by Golumbic [14] and an acyclic ordering by Bang-Jensen and Gutin [6]. We use this same ordering for the strong components of D: $D_1, \ldots, D_m$ so that v_i^* corresponds to component D_i.

Now, assume that C is a cycle of the underlying graph of D^*. Choose the smallest index i such that v_i^* is on C, and let v_j^* and v_k^* be the neighbors of v_i^* on this cycle. By the minimality of i we have $i < j$ and $i < k$, and without loss of generality we can assume that $j < k$. Since $v_i^* v_k^* \in E(C)$, the acyclic ordering requires that we must have $v_i^* v_k^* \in A(D^*)$. This in turn requires that there are distinct vertices x and y in D_i and D_k respectively such that $xy \in A(D)$. Now choose any vertex z in D_j.

Since D is arc-traceable, xy is on some hamiltonian path of D, and z must precede x or follow y on such a path, and so there must be either a path in D from z to x or a path in D from y to z. But clearly, any such path must contain an arc uv where $u \in D_p$ and $v \in D_q$ with $p > q$. This requires that $x_p^* x_q^* \in A(D^*)$, a contradiction. Hence, the underlying graph of D^* must be acyclic. □

Unfortunately, this necessary condition gives no information about strong digraphs, as the condensation of such a digraph is an isolated vertex, which is trivially a path.

Recently, Reid [34] began to study arc-traceable tournaments. Attempting to find conditions that are necessary and sufficient for a tournament to be arc-traceable and several related questions is the primary focus of this dissertation.

The necessary condition given above, along with the observation made earlier that the condensation of a tournament is a transitive tournament implies the following corollary.

Corollary 2.2. *If a tournament T is arc-traceable, then T has at most two strong components.*

As noted above, our necessary condition for digraphs gives no information about which strong tournaments are arc-traceable. However, we can use the hamiltonicity of strong tournaments to observe that the analysis of arc-traceable tournaments reduces to the study of arc-traceable strong tournaments.

Corollary 2.3. *A tournament T with two strong components is arc-traceable if and only if the subtournaments induced by the strong components are arc-traceable.*

Proof. Let T_1 and T_2 be the subtournaments induced by the strong components of T, and let every vertex of T_1 dominate each vertex in T_2. Now, every hamiltonian path of T must be of the form P_1P_2, with P_i a hamiltonian path of T_i for $i = 1, 2$. So, if T_i is not arc-traceable for $i = 1$ or $i = 2$, then there is some arc that is not part of any hamiltonian path of T.

Conversely, if both T_1 and T_2 are arc-traceable, then each arc within these subtournaments is also traceable in T. To conclude that T is arc-traceable, we must show that the remaining arcs are also traceable. Each of these arcs is of the form xy with $x \in V(T_1)$ and $y \in V(T_2)$. We use the hamiltonicity of each of these subtournaments to construct a hamiltonian path P_1 of T_1 that ends at x and a hamiltonian path P_2 of T_2 that begins at y. Consequently, P_1P_2 is a hamiltonian path of T containing xy, completing the proof. □

Thus, if we are able to obtain even partial results in the case of strong tournaments, these results will have simple extensions to all tournaments.

3. Upset tournaments

We begin by studying the class of upset tournaments. As noted in Chapter 1, these are precisely the set of tournaments obtained from transitive tournaments by reversing the arcs of any single path from the source (i.e., vertex of in-degree zero) to the sink (i.e. vertex of out-degree zero). This reversed path is referred to as the upset path. Reversing the arcs of a path leaves the scores of the vertices on the interior of the path unchanged, and the initial and terminal vertices of the path have their scores decreased and increased by one respectively. If we begin with a transitive n-tournament, this yields a tournament with score sequence $(1, 1, 2, 3, \ldots, n-3, n-2, n-2)$.

We note that an identical argument indicates that reversing any path from a vertex of out-degree one to a vertex of in-degree one in an upset tournament T yields a transitive tournament. This would seem to indicate that the upset path of an upset tournament may not be unique. To see that the upset path is never unique, let u and u' be the vertices of an upset tournament with out-degree one and let v and v' be vertices with in-degree one. Without loss of generality, assume that u dominates u' and v dominates v'. Thus, any path that originates at u must begin with the arc uu', and any path that terminates at v' must end with the arc vv'. So, if U is an upset path of the tournament that originates at u, then $U - u$ is a path from u' to a vertex of in-degree one and this shorter path is also an upset path. Similarly, if U terminates at v', then $U - v'$ is an upset path that terminates at v. On the other hand, if U is an upset path that originates at u', then u cannot be a vertex of U since u is not the terminal vertex of U and the only vertex that can follow u on a path is u'. In this case, uU is also an upset path of the tournament, and dually if U is an upset path ending at v, then Uv' is also an upset path.

To address the above ambiguity we can require that our upset path U contain the arcs uu' and vv' or, alternately, we can forbid the inclusion of these arcs. The former approach is used in [31] and [24], among others. In general, we prefer the latter approach; we require that U be as short as possible. However, if there is a hamiltonian path from a vertex of out-degree one to a vertex of in-degree one, it is often convenient to choose this hamiltonian path as U.

We present a result of Brualdi and Li [9] showing that upset tournaments are characterized by their score sequence. Their proof uses the adjacency matrices of upset tournaments. To illustrate the powerful inductive nature of this class of tournaments, we present an alternate proof.

Theorem 3.1 (Brualdi and Li [9]). *Let $n \geq 4$. An n-tournament T is an upset tournament if and only if the score sequence of T is* $(1, 1, 2, 3, \ldots, n-3, n-2, n-2)$.

Proof. Necessity follows by the argument above; that the reversal of the arcs of a path of the transitive tournament from source to sink leaves the scores of the vertices on the interior of the path unchanged, decreases the score of the source by 1 and increases the score of the sink by 1. This produces a tournament with the desired score sequence.

For sufficiency, we use induction. For $n = 4$, the tournament with score sequence $(1, 1, 2, 2)$ is unique and it is easily seen that this is an upset tournament. For $n > 4$, let T be a n-tournament with the given score sequence, let u and v be the vertices of T with score 1, and assume that $uv \in A$. Consider the tournament $T' = T - u$. Clearly, the score of every vertex x in T' is one less than the score of x in T, apart from v, whose score is unchanged. Thus, the score sequence of the $(n-1)$-tournament T' is $(1, 1, 2, 3, \ldots, n-4, n-3, n-3)$. By the induction hypothesis, T' is an upset tournament. Let U be an upset path of T'. The first vertex of U has out-degree 1 in T'. If v is on the path U, then we consider the subpath U' that begins at v and ends at the terminal vertex of U. Reversing this path in either T' or T leaves a transitive tournament, and hence U' is an upset path of both T' and T. If v is not on the path U, then the only neighbor of the initial vertex of U is clearly the second vertex of this path. Hence, v dominates the initial vertex of U. As a result, we can form the path $U' = vU$, and just as before, this path U' is an upset path of both T' and T. □

3.1 Arc-traceable upset tournaments

We now address the question of arc-traceability in upset tournaments. Initially, we characterize which arcs of an upset tournament lie on a hamiltonian path.

Theorem 3.2. *Let T denote an upset tournament on $n \geq 6$ vertices with upset path U. If V is labeled $V = \{v_1, \ldots, v_n\}$, so that $v_iv_j \in A$ if and only if either $i < j$ or v_iv_j is an arc of U, then the arc v_rv_s of T is non-traceable if and only if all of the following hold:*

- *Both $v_r \in V(U)$ and $v_s \in V(U)$.*
- *v_rv_s is not an arc of the upset path.*

- *For each vertex $v_k \in V(U)$ with $r < k < s$, neither v_{k-1} nor v_{k+1} are vertices of the upset path.*

Proof. Let $v_r v_s$ be an arc of T that is on no hamiltonian path. First, we show that both v_r and v_s are vertices of the upset path. This follows from the observation that $T - v_i$ is an upset tournament for any v_i not on the upset path. Since upset tournaments are strong, this tournament has a hamiltonian cycle and consequently, a hamiltonian path beginning or ending at any specified vertex. So if v_r is not on the upset path, we can choose a hamiltonian path H of $T - v_r$ that begins at the vertex v_s. Now, $v_r H$ is a hamiltonian path of T containing the arc $v_r v_s$, a contradiction. Similarly, if v_s is not on the upset path, we choose H, a hamiltonian path of $T - v_s$ that ends at the vertex v_r and $H v_s$ is a hamiltonian path containing $v_r v_s$. As no such path exists, $v_s \in V(U)$.

Next, we show that $v_r v_s$ is not an arc of the upset path. This follows from the observation that $T - V(U)$ is a transitive tournament whose source is dominated by v_1 in T. If H is the unique hamiltonian path of $T - V(U)$, UH is a hamiltonian path of T containing the arc $v_r v_s$. As no such path exists, $v_r v_s$ can not be an arc of the upset path. Thus, $r < s$ and for at least one k such that $r < k < s$, $v_k \in V(U)$.

Now, we show that for each vertex $v_k \in V(U)$ where $r < k < s$, neither v_{k-1} nor v_{k+1} is on the upset path. Suppose that for some such k, v_{k-1} is on the upset path, so $v_k v_{k-1}$ is an arc of U. Then, both $T_1 = T[\{v_1, \ldots, v_{k-1}\}]$ and $T_2 = T[\{v_k, \ldots, v_n\}]$ are upset tournaments and hence strong. We can then choose a hamiltonian path H_1 of T_1 that ends at v_r and a hamiltonian path H_2 of T_2 that begins at the vertex v_s. But then $H_1 H_2$ is then a hamiltonian path of

T containing v_rv_s, a contradiction. Similarly, if v_{k+1} is on the upset path then by the same argument with $T_1 = T[\{v_1, \ldots, v_k\}]$ and $T_2 = T[\{v_{k+1}, \ldots, v_n\}]$, v_rv_s is on a hamiltonian path. So, neither v_{k-1} nor v_{k+1} are on the upset path.

For the converse, suppose that v_rv_s is an arc of T, where both $v_r \in V(U)$ and $v_s \in V(U)$, v_rv_s is not an arc of the upset path, and that for each v_k on the upset path U between v_r and v_s, neither v_{k-1} nor v_{k+1} are on the upset path. Let Q be a longest path of T that contains the arc v_rv_s, and let $U[v_s, v_r]$ be the subpath of U beginning at v_s and ending at v_r.

First, note that Q can not contain $U[v_s, v_r]$ as a subpath, since v_s follows v_r on this path but v_r follows v_s on Q. So we can choose an upset arc v_qv_p of $U[v_s, v_r]$ that is not part of the path Q. Note, by assumption $p + 1 \neq q$, and every path from v_{p+1} to v_r must contain the arc v_qv_p. Since Q does not contain this arc, v_{p+1} does not precede v_r on Q. Similarly, every path from v_s to v_{p+1} must include the arc v_qv_p, and hence v_{p+1} does not follow v_s on Q. But v_r and v_s are consecutive on Q, and so v_{p+1} is not on the path Q. Thus, Q is not a hamiltonian path, and since the length of Q is maximal, no hamiltonian path containing v_rv_s exists. □

The above result completely characterizes arc-traceable upset tournaments with six or more vertices.

Corollary 3.3. *An upset tournament T on $n \geq 6$ vertices is arc-traceable if and only if for every vertex v_k on the interior of the upset path, either v_{k-1} or v_{k+1} is also on the upset path.*

Proof. If T is a tournament satisfying the given condition, then Theorem 3.2

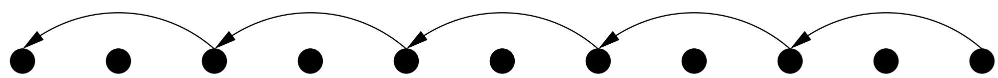


Figure 3.1: An 11-tournament with 10 non-traceable arcs

implies that there is no arc that is non-traceable, i.e. T is arc-traceable. For the converse, assume there is some v_k on the upset path with neither v_{k-1} nor v_{k+1} on the upset path. Let v_i and v_j be the vertices immediately preceding and succeeding v_k on U, respectively. Then, by Theorem 3.2 $v_j v_i$ is on no hamiltonian path. □

The above results fail for $n = 5$ vertices, as the upset tournament obtained from reversing the unique hamiltonian path of the transitive tournament on five vertices is not arc-traceable, despite satisfying the conditions of Corollary 3.3. This is a consequence of the fact that this tournament is isomorphic to the upset tournament with an upset path of length with middle vertex of score 2, and this tournament does not meet the criteria indicated in the corollary. In fact, there are only two non-isomorphic upset tournaments on five vertices. The other is obtained by reversing the single arc from source to sink in a transitive tournament, and this tournament is easily seen to be arc-traceable.

Next, we describe an upset tournament with many non-traceable arcs, and prove that this example is maximal.

Theorem 3.4. *If T is an upset tournament with $n \geq 5$ vertices, n odd, and the upset path of T is $v_n v_{n-2} v_{n-4} \ldots v_3 v_1$, then exactly $\frac{n^2-4n+3}{8} = \frac{1}{4} \cdot \binom{n}{2} - \frac{3(n-1)}{8}$ arcs of T are not on a hamiltonian path.*

Proof. By Theorem 3.2, the arcs of T on no hamiltonian path are of the form $v_i v_{i+(2k+2)}$ for i odd, $1 \le i \le n-4$ and $1 \le k \le \frac{n-1}{2} - 1$. Thus, for a fixed $i = 2j+1$, there are exactly $\frac{n-i}{2} - 1 = \frac{n-(2j+1)}{2} - 1 = \frac{n-3}{2} - j$ non-traceable arcs starting at vertex v_i. Summing all possible values of j, we obtain

$$\sum_{j=0}^{\frac{n-5}{2}} \left(\frac{n-3}{2} - j\right) = \left(\frac{n-3}{2}\right) \cdot \left(\frac{n-3}{2}\right) - \frac{\frac{n-5}{2}\frac{n-3}{2}}{2} = \frac{n^2-4n+3}{8}$$

non-traceable arcs in T. □

Figure 3.1 shows a tournament of order 11 with the structure described above. We now show that this family of examples has the maximal number of non-traceable arcs among all upset n-tournaments.

Theorem 3.5. *An upset n-tournament T, $n \ge 5$, has at most*

$$\frac{n^2-4n+3}{8} = \frac{1}{4} \cdot \binom{n}{2} - \frac{3(n-1)}{8}$$

non-traceable arcs.

Proof. We prove the result by induction on n. For $n = 5$, it is easy to verify that any upset tournament on five vertices not having the structure indicated in Theorem 3.4 is arc-traceable, and that the upset tournament with this structure has $\frac{5^2-4\cdot 5+3}{8} = 1$ arc that is not on any hamiltonian path.

Next, assume the result for upset tournaments with fewer than $n > 5$ vertices. Let i be the vertex that immediately precedes the vertex v_1 on the upset path. As observed earlier, $T_1 = T[v_i, \ldots, v_n]$ is an upset tournament, and so by the induction hypothesis, there are at most

$$\frac{(n-i+1)^2 - 4(n-i+1) + 3}{8} = \frac{n^2-4n+3}{8} - (i-1)\frac{2n-4-(i-1)}{8}$$

non-traceable arcs in T_1. By applying Theorem 3.2 twice, once for necessity in T_1 and again for sufficiency in T, we note that each of these arcs is also non-traceable in T.

All that remains is to count the non-traceable arcs of T that are not arcs of T_1. Clearly, no arc incident with the vertices $v_2, \ldots, v_{i-1}$ is non-traceable so any non-traceable arc of T that is not an arc of T_1 is incident with v_1. Additionally, if $i = 2$, then by Theorem 3.2, every arc incident with v_1 is also on a hamiltonian path. In this case, the Theorem follows by observing that $\big(i-1\big)\frac{2n-4-(i-1)}{8} = \frac{2n-5}{8} > 0$ as $n > 5$. So, we may assume that $i \geq 3$. In this case, as $n > 5$ it follows that

$$\frac{(n-i+1)^2 - 4(n-i+1) + 3}{8} \leq \frac{(n-2)^2 - 4(n-2) + 3}{8} = \frac{n^2 - 8n + 15}{8}.$$

Thus, we must show that at most $\frac{4n-12}{8} = \frac{n-3}{2}$ arcs incident with v_1 are non-traceable. Let j be the largest index such that the arc v_1v_j is non-traceable. As a consequence of Theorem 3.2, at most $\frac{j-1}{2}$ of the vertices $v_1, \cdots v_j$ are on the upset path of T and v_i is one of these vertices. Since v_iv_1 is part of the upset path, and thus is part of a hamiltonian path, this implies that at most $\frac{j-1}{2} - 1 \leq \frac{n-3}{2}$ arcs incident with the vertex v_1 are not a part of any hamiltonian path, and the result follows. □

3.2 Counting Hamiltonian Paths in Upset Tournaments

To conclude this section, we consider the connection between arc-traceability and the number of distinct hamiltonian paths in upset tournaments. We note a result by Moon [28] that gives a lower bound for the number of hamiltonian paths in strong tournaments.

Theorem K (Moon [28]). *An strong tournament on n vertices has at least $6^{\frac{n-1}{4}} \approx (.639)(1.565^n)$ distinct hamiltonian paths.*

It is natural to expect that, in general, arc-traceable upset tournaments have more hamiltonian paths than non-arc-traceable upset tournaments. However, we shall now show that this is not always the case.

We observe again that for any vertex v in an upset tournament T, $T - v$ is also an upset tournament if v is not the interior of some upset path. If v is on the interior of some upset path, then it is a cut vertex and the first and last strong components of $T - v$ are upset tournaments with the remaining strong components all trivial (consisting of single vertices). Combined with the observation that the number of hamiltonian paths in a tournament T is the product of the number of hamiltonian paths of the strong components of T, this suggests that we can find the number of hamiltonian paths of an upset tournament recursively.

In particular, we next consider two simple techniques that will show that it suffices to count the number of hamiltonian paths of any upset tournament in which no three consecutive vertices lie on the upset path. Let p_T denote the number of distinct hamiltonian paths of the tournament T.

Theorem 3.6. *If T is an upset n-tournament and v_j is a vertex on the upset path such that neither v_{j-1} nor v_{j+1} are on the upset path, then then $p_T = p_{T[v_1,v_j]}p_{T[v_j,v_n]}$.*

Proof. Let $\mathcal{H}$ be the set of hamiltonian paths of T, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the sets of hamiltonian paths of $T[v_1, \ldots, v_j]$ and $T[v_j, \ldots, v_n]$, respectively. We show a bijection between $\mathcal{H}$ and $\mathcal{H}_1 \times \mathcal{H}_2$.

Let H be a hamiltonian path of T. We define the sequence of vertices S_1 by deleting each vertex of H with index greater than j. We claim that S_1 is a hamiltonian path of $T[v_1, \ldots, v_j]$. Assume otherwise and let v_s and v_t be consecutive in S_1 with $v_s \not\rightarrow v_t$. Since H is a path, v_s and v_t can not be consecutive on H, and so there must exist a vertex v_r between v_s and v_t on H with $r > j$. But since T is an upset tournament every path from v_r to v_t must include the vertex v_j, and since v_j was not deleted from S_1, we conclude that $t = j$ and so $s < t$. Since $v_s v_t$ is not an arc of T, $v_t v_s$ must be an upset arc. Since v_s precedes v_t on H, this arc is not part of H. It follows that this arc is included on every path from v_{j-1} to v_s in T as well as on every path from v_t to v_{j-1}. Thus, we conclude that v_{j-1} does not precede v_s or succeed v_t on H. Since H is a hamiltonian path, this vertex must lie between v_s and v_t on this path, contradicting the fact that these vertices are consecutive in S_1. Consequently, S_1 must be a hamiltonian path of $T[v_1, \ldots, v_j]$. Similarly, the sequence S_2 obtained by deleting the vertices of H with index less than j is a hamiltonian path of $T[v_j, \ldots, v_n]$.

On the other hand, let $P_1 v_j Q_1$ and $P_2 v_j Q_2$ be hamiltonian paths of $T[v_1, \ldots, v_j]$ and $T[v_j, \ldots, v_n]$, respectively. Then $H = P_1 P_2 v Q_1 Q_2$ is a hamiltonian path of T. □

Our next result is a similar result that applies to tournaments in which two consecutive vertices may be on the upset path.

Theorem 3.7. *Let T be an upset tournament, and let $v_{i+1} v_i$ be an arc of the upset path of T, where v_{i+2} and v_{i-1} are not vertices of the upset path. Then* *the number of hamiltonian paths of T is $2ab - cd$ where a is $p_{T[v_1,\ldots,v_i]}$, b is $p_{T[v_{i+1},\ldots,v_n]}$, c is $p_{T[v_{i+2},\ldots,v_n]}$ and d is $p_{T[v_1,\ldots,v_{i-1}]}$.*

Proof. The proof is similar to the proof of Theorem 3.6. Begin with $P_1 v_i Q_1$ and $P_2 v_{i+1} Q_2$, hamiltonian paths of $T[v_1, \ldots, v_i]$ and $T[v_{i+1}, \ldots, v_n]$, respectively. Now, define two sequences of vertices $H_1 = P_1 P_2 v_{i+1} v_i Q_1 Q_2$ and $H_2 = P_1 v_i Q_1 P_2 v_{i+1} Q_2$. Clearly, H_1 is a hamiltonian path and H_2 is as well provided that either $Q_1 \neq \emptyset$ or $P_2 \neq \emptyset$. Now if $Q_1 = \emptyset$, then P_1 must be a hamiltonian path of $T[v_1, \ldots, v_{i-1}]$ and if $P_2 = \emptyset$ then Q_2 is a hamiltonian path of $T[v_{i+2}, \ldots, v_n]$. Thus the number of distinct hamiltonian paths of T is at least $ab + (ab - cd) = 2ab - cd$ where a, b, c and d are defined as in the Theorem.

For the reverse inequality, let H be a hamiltonian path of T. First, assume that $v_{i+1}v_i$ is an arc of H. Then let $H = P v_{i+1} v_i Q$ and let P_1 and Q_1 be the vertices of P and Q, respectively, with index less than i. We claim that $P_1 v_i Q_1$ is hamiltonian path of $T[v_1, \ldots, v_i]$. Assume otherwise. Then, there are consecutive vertices v_s and v_t in the sequence $P_1 v_i Q_1$ such that v_t dominates v_s. This requires that v_{i+1} must succeed v_s and precede v_t on H, and thus $t = i$. Since v_i dominates only one vertex in $T[v_1, \ldots, v_i]$, this vertex must be v_s, and since v_{i-1} is not on the upset path $s < i - 1$. As above, the arc $v_t v_s$ is on every path from v_t to v_{i-1} as well as every path from v_{i-1} to v_s. Since this arc is not included in H, v_{i-1} must precede v_t and succeed v_s. This contradicts our choice of v_s and v_t as consecutive vertices of $P_1 v_i Q_1$, so this sequence of vertices forms a hamiltonian path of $T[v_1, \ldots, v_i]$. Similarly, $P_2 v_{i+1} Q_2$ is a hamiltonian path of $T[v_{i+1}, \ldots, v_n]$ where $P_2 = P \setminus P_1$ and $Q_2 = Q \setminus Q_1$. Thus, there are at most ab hamiltonian paths of T that contain the arc $v_{i+1}v_i$.

Now, assume that $v_{i+1}v_i$ is not an arc of H. Then H is also a hamiltonian path of the tournament T' obtained by reversing this arc. T' has two strong components, $T[v_1, \ldots, v_i]$ and $T[v_{i+1}, \ldots, v_n]$ and so any hamiltonian path of T' must have the form PQ where P is a hamiltonian path of $T[v_1, \ldots, v_i]$ and Q is a hamiltonian path of $T[v_{i+1}, \ldots, v_n]$. So H has this form and since H is also a hamiltonian path of T, we cannot have both P ending at v_i and Q beginning at v_{i+1}. Clearly, there are ab ways of choosing an arbitrary P and Q, and there are cd ways of choosing such that both P ends at v_i and Q begins at v_{i+1}. Thus, there are at most $ab - cd$ hamiltonian paths of T that do not contain the arc $v_{i+1}v_i$. □

Next, we count the hamiltonian paths of an upset tournament with an upset path of length one. This, combined with the previous two results, permits a recursive technique that can be used to determine the number of hamiltonian paths of any upset tournament in which no three consecutive vertices are on the upset path.

Theorem 3.8. *The upset tournament on n vertices with a single upset arc has $2^{n-2} + 1$ hamiltonian paths.*

Proof. Let T be an upset n-tournament with the single upset arc v_nv_1.

First, we consider hamiltonian paths that do not use the upset arc. Each such path is also a hamiltonian path of the transitive tournament obtained by reversing this upset arc. Since the transitive tournament has a unique hamiltonian path, we conclude that there is a unique hamiltonian path of T not using the arc v_nv_1.

Now, we consider hamiltonian paths using the upset arc of T. Let S_1 be the set of vertices preceding v_n on such a hamiltonian path. We note that S_1 is a subset of $S = \{v_2, v_3, \ldots, v_{n-1}\}$. Conversely, let S_1 be any subset of S and let $S_2 = S \setminus S_1$. Since $T[S_1]$ and $T[S_2]$ are both transitive tournaments, and v_n is dominated by S_1 and v_1 dominates S_2, there is a unique hamiltonian path of T containing the arc $v_n v_1$ where S_1 is the set of vertices preceding v_n. Thus, there is a bijection between subsets of S and hamiltonian paths of T containing the arc $v_n v_1$ and consequently, it follows that there are exactly $2^{|S|} = 2^{n-2}$ hamiltonian paths containing the arc $v_n v_1$ and $2^{n-2} + 1$ hamiltonian paths in total. □

Corollary 3.9. *If T is an upset tournament on n vertices with an upset path of length 2, then T has at least $2^{n-3} + 2^{\frac{n-1}{2}} + 1$ hamiltonian paths, with equality if and only if n is odd and the upset path is $v_n v_{\frac{n+1}{2}} v_1$.*

Proof. Let v_j be the middle vertex of the upset path. Then by Theorem 3.6, $p_T = p_{T[v_1,\ldots,v_j]} p_{T[v_j,\ldots,v_n]}$. Furthermore, by Theorem 3.8 $p_{T[v_1,\ldots,v_j]} = 2^{j-2} + 1$ and $p_{T[v_j,\ldots,v_n]} = 2^{n-j-1} + 1$. Thus $p_T = 2^{n-3} + 2^{j-2} + 2^{n-j-1} + 1$. The proof is complete by noting that the function $f(x) = 2^{x-2} + 2^{n-1-x}$ has a global minimum at $x = \frac{n+1}{2}$. □

We note, as a consequence of this corollary, the number of hamiltonian paths in upset n-tournaments with upset paths of length two grows exponentially with n at a faster rate than the lower bound given by Theorem K. Specifically, $\frac{1}{8}\, 2^n < p_T < \frac{1}{4}\, 2^n$ for $n > 5$. Additionally, we note that for $n \geq 5$, upset tournaments with upset paths of length two are necessarily non-arc-traceable by Corollary 3.3.

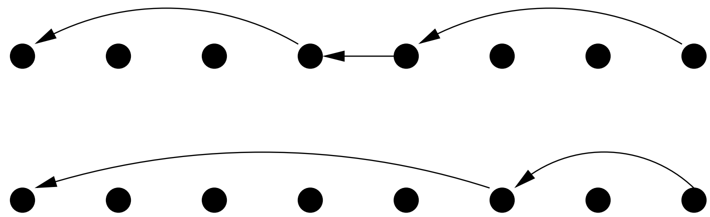


Figure 3.2: Two 8-tournaments

We can now use the above results to show that the first 8-tournament shown in Figure 3.2 has precisely $2(5)(5) - 1 = 49$ hamiltonian paths and by Corollary 3.3 is arc-traceable. Similarly, the second 8-tournament shown in Figure 3.2 is non-arc-traceable and yet has $3(17) = 51$ hamiltonian paths.

The above results also permit us to count the number of hamiltonian paths of every upset tournament on seven or fewer vertices, with one exception. The only exception is the upset tournament on seven vertices with upset path $v_7v_5v_4v_3v_1$, which is isomorphic to the upset tournament with upset path $v_7v_6v_5v_4v_3v_2v_1$. For this case, we will show that this tournament has 31 distinct hamiltonian paths. Table 3.1 lists all upset tournaments of order three through seven, the number of distinct hamiltonian paths, and whether the tournament is arc-traceable or not according to Corollary 3.3. From the data, we conclude that the tournaments in Figure 3.2 are minimal with respect to the number of vertices. In other words, if T is an arc-traceable upset n-tournament with k hamiltonian paths, and T' is a non-arc-traceable upset n-tournament with $k' > k$ hamiltonian paths, then $n \geq 8$.

We now consider the family of upset n-tournaments with upset paths of length $n-1$. In other words, the tournaments obtained by reversing the arcs in

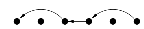
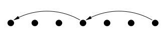
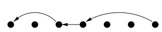
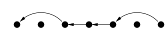
Tournament: T	p_T	Arc-traceable?	Tournament: T	p_T	Arc-traceable?
	3	Yes		5	Yes
	9	Yes		9	No
	17	Yes		15	No
	15	No		17	Yes
	33	Yes		27	No
	25	No		29	Yes
	27	No		27	No
	29	Yes		31	Yes

Table 3.1: All upset tournaments of order three through seven.

the unique hamiltonian path of a transitive n-tournament. We show that the the number of hamiltonian paths of such an n-tournament satisfies the "tribonacci" recurrence for $n \geq 3$ and use this recurrence to obtain asymptotic results for the number of hamiltonian paths in such tournaments. For convenience, in the remainder of this chapter we will use the notation T_i to refer to the upset tournament with i-vertices with an upset path of length $i-1$. We will then denote the number of hamiltonian paths of T_i as p_i.

In the following proof, we require some new notation. We will say that a path P is v-*deletable* when v is on the path P, and either v is the initial vertex of P or the vertex immediately preceding v dominates the vertex immediately following v. Conversely, v can be *inserted* into P if v is not on P and there are consecutive vertices x and y on P such that $x \to v$ and $v \to y$. If the terminal vertex of P dominates v, then we will say that v can be *appended* onto P. Dually, if the initial vertex of P is dominated by v, then we will say that v can be *prepended* onto P.

Theorem 3.10. *$p_n = p_{n-1} + p_{n-2} + p_{n-3}$ for $n \geq 3$.*

Proof. Let S_i be the set of hamiltonian paths of T_i. We establish the result by showing a bijection between S_n and $S_{n-1} \cup S_{n-2} \cup S_{n-3}$.

Initially, we partition the set S_n into two parts; $S_{n,1}$, the hamiltonian paths ending at v_n and $S_{n,2}$, those ending at any vertex other than v_n. We further subdivide each $S_{n,i}$; let $U_{n,1}$ be the set of all hamiltonian paths of $S_{n,1}$ that are v_{n-1} deletable and let $U_{n,2}$ be the set of all hamiltonian paths of $S_{n,2}$ that are v_n deletable. Finally, let $W_{n,i} = S_{n,i} \setminus U_{n,i}$ for $i = \{1, 2\}$.

We now construct the desired bijection in three parts. First, we note that there is a bijection between $U_{n,2}$ and S_{n-1}. By deleting v_n from each path of $U_{n,2}$, we obtain a hamiltonian path of T_{n-1} and since $N^+(v_n) = \{v_{n-1}\}$, v_n can be inserted into or prepended onto a hamiltonian path of T_{n-1} in a unique way. Since no path of $U_{n,2}$ ends at v_n, v_n is interior on every path of $U_{n,2}$ and thus two hamiltonian paths of $U_{n,2}$ are distinct if and only if they can be obtained in this way from distinct hamiltonian paths of T_{n-1}.

Next, we observe a bijection between $U_{n,1}$ and S_{n-2}. By deleting both v_n and v_{n-1} from a hamiltonian path of $U_{n,1}$, we obtain a hamiltonian path of T_{n-2}, and as in the previous case, v_{n-1} can be inserted into the interior of each hamiltonian path of T_{n-2} in a unique way. Furthermore, we can append v_n to the end of each of these paths, thus obtaining hamiltonian paths of $U_{n,1}$. Just as above, two paths of $U_{n,1}$ are distinct if and only if they can be obtained in this way from distinct paths of T_{n-2}.

The last part of our bijection is between $W_{n,1} \cup W_{n,2}$ and S_{n-3}. We first show that each path P of $W_{n,1}$ must end with the subpath $v_{n-3}v_{n-1}v_{n-2}v_n$. Since a path P of $W_{n,1}$ does not end at v_{n-1}, and $N^+(v_{n-1}) = \{v_{n-2}\}$, P must include the arc $v_{n-1}v_{n-2}$. Furthermore, because P is not v_{n-1} deletable, the vertex immediately preceding v_{n-1} must be dominated by v_{n-2}, and so must be a vertex in $N^+(v_{n-2}) = \{v_{n-3}, v_n\}$. As P ends at v_n by definition, this vertex must be v_{n-3} and the vertex immediately following v_{n-2} must be v_n, the terminal vertex of P.

Similarly, we show that each path P of $W_{n,2}$ must end with the subpath $v_{n-2}v_nv_{n-1}$. Since P does not end at v_n and $N^+(v_n) = \{v_{n-1}\}$, it contains the arc v_nv_{n-1} and since P is not v_n deletable, the vertex immediately preceding v_n on P must be in the set $N^+(v_{n-1}) = \{v_{n-2}\}$. Further, since this is the only vertex dominated by v_{n-1}, v_{n-1} must be the terminal vertex of P. Thus, for any path of $W_{n,1} \cup W_{n,2}$ the first $n-3$ vertices form a hamiltonian path of T_{n-3}. Conversely, any hamiltonian path H of T_{n-3} can be extended to a hamiltonian path of T_n by appending the path $v_{n-1}v_{n-2}v_n$ if H ends at v_{n-3}, or appending the path $v_{n-2}v_nv_{n-1}$ if H does not end at v_{n-3}. □

Next, we note that $p_1 = p_2 = 1$ and $p_3 = 3$, and this defines the sequence A000213 in the database maintained by Sloane [37] (offset by one). Note that by Corollary 3.3, T_i is not arc-traceable for all $i \geq 6$. Also, $p_7 = 31$, completing our table of the number of distinct hamiltonian paths for all upset tournaments with between three and seven vertices.

The tribonacci recurrence shares many similarities with the better known Fibonacci recurrence. In particular, just as the ratio of Fibonacci numbers approaches the positive real root of $x^2 - x - 1$, the ratio of consecutive numbers in a sequence satisfying the tribonacci recurrence approaches the positive (and only) real root of $x^3 - x^2 - x - 1$ [22].

Corollary 3.11. *The number of hamiltonian paths of an upset n-tournament with upset path of length $n-1$ approaches α^n as n approaches infinity, where α is the real root of $x^3 - x^2 - x - 1$ (approximately 1.8329).*

Thus, it is clear that the tournaments in Figure 3.2 are not isolated examples; in fact, there exist infinitely many such examples. For $n \geq 15$, the non-arc-traceable upset tournament T with upset path $v_n v_{\lceil \frac{n+1}{2} \rceil} v_1$ has more distinct hamiltonian paths than the arc-traceable upset tournament T' with upset path $v_n v_{n-1} \ldots v_1$, and $p_T - p_{T'}$ grows exponentially with n.

4. Arc-traceability in all strong tournaments

In this chapter we turn our attention to arc-traceability in all strong tournaments. We require very different techniques than were used for upset tournaments, and we will fall just short of a characterization of arc-traceable strong tournaments. However, we will develop a specific structure that is shared by all non-arc-traceable strong tournaments, and we will use this structure to obtain many sufficient conditions for strong tournaments to be arc-traceable and to yield some extremal results for non-arc-traceable tournaments.

4.1 Observations on arc-traceability in tournaments

We recall that in a hamiltonian digraph D, then for any $v \in V(D)$, D has a hamiltonian path that begins at v and also a hamiltonian path that ends at v. Since every strong tournament is hamiltonian, we observe that non-traceable arcs of a tournament must join two cut vertices.

Proposition 4.1. *If T is a strong tournament and v is not a cut-vertex of T, then every arc incident with v is on a hamiltonian path.*

This gives us our first sufficient condition for arc-traceable tournaments; if T has no cut vertices, then T has no non-traceable arcs.

Corollary 4.2. *If T is 2-connected, then T is arc-traceable.*

This first sufficient condition was observed in the more general setting of multipartite tournaments by Volkmann in [42]. Our second sufficient condition requires the following result, using a technique similar to the proof of Theorem H by Moon in [29].

Lemma 4.3. *Let xy be an arc in a strong tournament T. If P is a longest path from y to x in T, then P can be extended to a hamiltonian cycle.*

Proof. Let P be a longest path from y to x and let C be the longest cycle that contains P. Assume that C is not hamiltonian. Thus, there is some vertex v that is not on C. If y dominates v, then v must dominate each vertex of P. Otherwise, let u be the first vertex along P such that $vu \in A(T)$ and let w be the vertex immediately preceding u on P. Now, $y \ldots wvu \ldots x$ is path from y to x longer than P. Thus, v dominates x and by a similar argument using the maximality of C, v must also dominate the subpath of C from x to y. In other words, v dominates all of the vertices of C. But since T is strong, there must be a path from every vertex of C to v. Let Q be the shortest of these paths and note that this minimality requires that the only vertex of Q on C is the initial vertex v_q. Let v_q^- and v_q^+ be the immediate predecessor and successor to v_q on C, respectively. If the arc $v_q v_q^+$ is an arc of P, then $y \ldots v_q^- Q v_q^+ \ldots x$ is a path from y to x longer than P. Otherwise, the cycle C' obtained by replacing the arc $v_q v_q^+$ with the path $Q v_q^+$ is a cycle containing P that is longer than C. In either case, we have obtained a contradiction. □

We now can show:

Theorem 4.4. *Let T be a strong n-tournament, and let $xy \in A(T)$ be on some cycle of length $l > \frac{n+1}{2}$. Then xy is on a hamiltonian path of T.*

Proof. Let $C = yu_1 \cdots u_k x$ be the longest cycle containing the arc xy. Extend $C - xy$ to H, a hamiltonian cycle of T. Let $H = yu_1 \cdots u_k x w_m \cdots w_1$. So $n = k + m + 2$, and since C has length $k + 2 > \frac{n+1}{2}$, $k \geq m$.

If w_i dominates u_i for some i, choose the minimal such i. If $i = 1$ then define the path $P = w_m \cdots w_1 u_1 \cdots u_k xy$, otherwise define the path $P = w_m \cdots w_i u_i \cdots u_k x y u_1 \cdots u_{i-1} w_{i-1} \cdots w_1$. In either case, P is a hamiltonian path of T containing xy. So we may assume that u_i dominates w_i for every $i, 1 \leq i \leq m$. In particular, we may assume that $u_m w_m \in A(T)$. If $k = m$, then let $P = xyu_1 \cdots u_k w_m \cdots w_1$, and if $k > m$, set $P = u_{m+1} \cdots u_k x y u_1 \cdots u_m w_m \cdots w_1$. In either case, P is a hamiltonian path of T containing the arc xy. □

From this we infer a second sufficient condition for a strong tournament to be arc-traceable.

Corollary 4.5. *If T is a strong tournament and every arc of T is on some cycle of length $l > \frac{n+1}{2}$ then T is arc-traceable.*

We now consider again our observation above that every 2-connected tournament is arc-traceable. If a tournament is two-connected (and hence arc-traceable), then Theorem E states that for any two vertices x and y such that $y \not\to x$, there must be at least two internally vertex disjoint paths from y to x. Of course, in a tournament $y \not\to x$ implies that $x \to y$. So we can re-state our observation above as follows.

Proposition 4.6. *If T is a tournament with the property that there exist at least two internally vertex disjoint path from y to x for every arc xy, then T is* *arc-traceable.*

We show this property applies locally as well. Any arc xy where the smallest y, x separating set has size at least 2 is traceable. We use Theorem E and state the result in terms of internally vertex disjoint paths.

Theorem 4.7. *Let T be a strong tournament, and let $xy \in A(T)$ be such that there exist at least two internally vertex disjoint paths from y to x in T. Then there exists a hamiltonian path of T which includes the arc xy.*

Proof. Let P and Q be two internally vertex disjoint paths from y to x such that for any internally vertex disjoint paths P', Q' from y to x, $|V(P) \cup V(Q)| \geq |V(P') \cup V(Q')|$. Let $P = yu_0 \cdots u_m x$ and $Q = yw_0 w_1 \cdots w_k x$, and let $U = \{u_0, \ldots, u_m\}$ and $W = \{w_0, \ldots, w_k\}$. If $V(P) \cup V(Q) = V(T)$, then $H = u_0 \cdots u_m xyw_1 \cdots w_k$ is a hamiltonian path of T containing xy. So assume $V(P) \cup V(Q) \subset V(T)$ and let S be the tournament induced on the vertices of $V(T) \setminus (V(P) \cup V(Q))$. If $V(S)$ dominates $U \cup W$, then for any hamiltonian path H_S of S, $H = H_S u_0 \cdots u_m xyw_0 \cdots w_k$ is a hamiltonian path of T containing xy. Additionally, if $V(S)$ is dominated by $U \cup W$, then $H = u_0 \cdots u_m xyw_0 \cdots w_k H_S$ is a hamiltonian path of T which contains xy for any hamiltonian path H_S of S. Hence, we may assume that $V(S)$ neither dominates nor is dominated by $U \cup W$. Choose the minimal r such that a vertex v of the r^{th} strong component of S is beaten by some vertex of $U \cup W$. Without loss of generality, assume that $u_i v \in A(T)$ for some i.

If $vu_j \in A(T)$ for some $j > i$, choose the minimal such j. Observe that $P' = yu_0 \cdots u_{j-1} v u_j \cdots u_m x$ and $Q' = Q$ are internally vertex disjoint paths from

y to x with $|V(P')\cup V(Q')| > |V(P)\cup V(Q)|$, contradicting the maximality of P and Q. Consequently, u_j dominates v for every $j > i$, and in particular, $u_m v \in A(T)$. Now, partition S into two disjoint subtournaments, S_1 (possibly empty) containing the vertices of the first $r-1$ components of S and S_2 containing the remaining vertices. Thus, v is in the initial strong component of the tournament S_2. Let H_1 be any hamiltonian path of S_1 and let H_2 be a hamiltonian path of S_2 that begins at v. Since the terminal vertex of H_1 is from the $(r-1)^{\text{st}}$ component of S, this vertex dominates $U \cup W$ by the minimality of r. Thus, $H = H_1 w_0 \cdots w_k x y u_0 \cdots u_m H_2$ is a hamiltonian path of T containing xy. □

The above result can be used to derive some tests for traceability for a given arc xy. In the next section we will show that each of the following results can be improved.

Corollary 4.8. *Let xy be an arc of T. If $d^+(y) > d^+(x)$, then there is a hamiltonian path of T that includes the arc xy.*

Proof. As $d^+(y) > d^+(x)$, $|N^+(y)| > |N^+(x)|$. Furthermore, $xy \in A(T)$ so $y \in N^+(x) \setminus N^+(y)$. Thus

$$
\begin{aligned}
|N^+(y) \setminus N^+(x)| &= |N^+(y)| - |N^+(y) \cap N^+(x)| \\
&\geq |N^+(y)| - |N^+(x) \setminus \{y\}| = d^+(y) - (d^+(x) - 1) \\
&\geq 2.
\end{aligned}
$$

Thus, there exist distinct vertices u_1 and u_2 such that yu_ix is a path of length two for $i = 1, 2$, and by Theorem 4.7, the arc xy is traceable. □

The next three sufficient conditions for arc-traceable tournaments are closely related, and follow from the fact that all 2-connected tournaments are arc-traceable, which can be thought of as a corollary of Theorem 4.7.

Corollary 4.9. *If T is a tournament with $\delta^0 > \frac{n+1}{4}$, then T is arc-traceable.*

Proof. We prove that the condition guarantees that T is 2-connected. Assume otherwise, and choose v such that $T-v$ is not strong. Let a and b be the number of vertices in the initial and terminal strong components of $T-v$, respectively. Then $T-v$ must contain a vertex with in-degree at most $\frac{a-1}{2}$ and a vertex with out-degree at most $\frac{b-1}{2}$. Thus,

$$2\delta^0(T) \leq \delta^+(T)+\delta^-(T) \leq \delta^+(T-v)+\delta^-(T-v)+2 \leq \frac{a+b-2}{2}+2 = \frac{a+b+2}{2}.$$

Finally, since $n \geq a+b+1$, $\delta^0(T) \leq \frac{n+1}{4}$. □

Corollary 4.10. *If T is a tournament such that for every $uv \in A(T)$, $d^-(u) + d^+(v) > \frac{n+1}{2}$, then T is arc-traceable.*

Proof. The proof is the same as for Corollary 4.9, combined with the additional observation that the vertex u with in-degree at most $\frac{a-1}{2}$ in $T-v$ dominates the vertex w with out-degree at most $\frac{b-1}{2}$ in $T-v$. Thus we have an arc uw such that

$$d^-(u)+d^+(w) \leq d^-_{T-v}(u)+d^+_{T-v}(w)+2 \leq \frac{a+b-2}{2}+2 \leq \frac{n+1}{2}.$$

□

Corollary 4.11. *Let T be a tournament such that $i(T) < \frac{n-3}{4}$. Then T is arc-traceable.*

Proof. The result follows from Corollary 4.9 and the fact that $i(T) = (n-1) - 2\delta^0$. Thus, $\delta^0 > \frac{n+1}{4}$ implies that $i(T) < (n-1) - \frac{n+1}{2} = \frac{n-3}{4}$. □

4.2 The structure of non-arc-traceable tournaments

Since the results of the previous section are really based on the observation that every two-connected tournament is arc-traceable, Theorem 4.7 has not yet proven useful in the study of arc-traceable tournaments. However, the contrapositive of Theorem 4.7 can be used to obtain some important necessary conditions for non-arc-traceable tournaments. This necessary structure is the basis for improving the bounds given in the Corollaries above and for many of the results to follow.

Theorem 4.12. *If T is a strong tournament, and $xy \in A(T)$ is not on a hamiltonian path, then:*

(i) There exists a vertex z such that $T - z$ is not strong.

(ii) $T - z$ has k strong components, $k \geq 4$.

(iii) x is in the initial strong component of $T - z$, and y is in the terminal strong component of $T - z$.

(iv) z is dominated by the 2^{nd} strong component of $T - z$ and z dominates the $(k-1)^{st}$ strong component of $T - z$.

Proof. By Theorem 4.7, there is at most a single vertex disjoint path from y to x in T, and since T is strong, there must be exactly one such path. Equivalently, by Theorem E, there is a y, x-separating set of size one. Let $\{z\}$ be this set.

Thus, there is no path from y to x in $T - z$, so y and x are in different strong components of $T - z$ and this tournament is not strong, establishing (i). Let T_z denote $T - z$, k be the number of strong components of T_z, and $T_z^{(1)}, T_z^{(2)}, \ldots T_z^{(k)}$ be the strong components of T_z where $T_z^{(i)}$ dominates $T_z^{(j)}$ whenever $i < j$.

As x dominates y in T_z, x must be in a strong component that precedes the strong component containing y. So, x can not be in the terminal strong component of T_z. If x is not in the initial strong component of T_z, then $T - x$ is strong and every arc incident with x is on some hamiltonian path. Thus, x must be in the initial strong component of T_z. Similarly, y must be a cut vertex of T, so y must be in the terminal strong component of T_z, establishing (iii).

If $k = 2$, then let P_1 be a hamiltonian path of $T_z^{(1)}$ that terminates at x, and P_2 be a hamiltonian path of $T_z^{(2)}$ that begins at y. Since T is strong, z must beat some vertex of the path P_1. Let w be the first such vertex along the path P_1. If w is the initial vertex of P_1, then let $H = zP_1P_2$. Otherwise, let w^- be the vertex immediately preceding w on P_1 and replace the arc w^-w of P_1 with the path w^-zw to form P_1' and let $H = P_1'P_2$. In either case, H is a hamiltonian path of T containing xy, contradicting the choice of xy. So, $k \geq 3$.

Next, assume that there is some $z^+ \in N_T^+(z) \cap V(T_z^{(2)})$. Since T is strong, we can choose $z^- \in N_T^-(z) \cap V(T_z^{(k)})$, and let P_k be some path from y to z^- in $T_z^{(k)}$. Now, let P_k' be a hamiltonian path of the tournament induced on $V(T_z^{(k)}) \setminus V(P_k)$, let P_1 be a hamiltonian path of $T_z^{(1)}$ that terminates at x, let P_2 a hamiltonian path of $T_z^{(2)}$ that originates at z^+, and let P_i be any hamiltonian path of $T_z^{(i)}$ for $3 \leq i \leq k-1$. Combining these paths we can construct $H = P_1P_kzP_2P_3 \cdots P_{k-1}P_k'$, a hamiltonian path of T containing xy. Hence, there

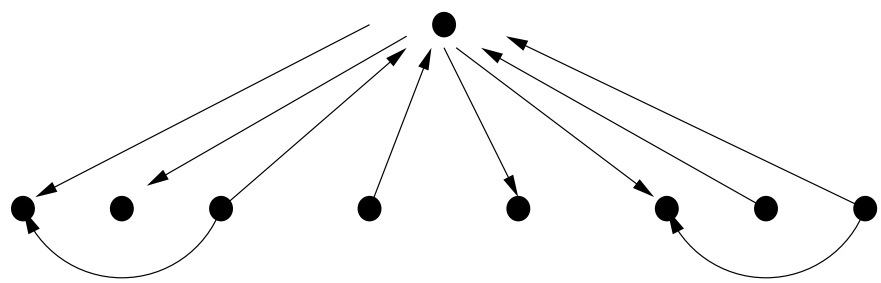


Figure 4.1: An arc-traceable tournament.

is no such z^+ in $V(T_z^{(2)})$. Dually, we have that there is no $N_T^-(z) \cap V(T_z^{(k-1)}) = \emptyset$, establishing (iv). This argument also shows that $k-1 \neq 2$, and so $k \neq 3$. Hence, $k \geq 4$ establishing (ii). □

The above result indicates a necessary structure for tournaments that are not arc-traceable. However, such a structure is not sufficient to guarantee non-arc-traceability. For example, the arc-traceable tournament in Figure 4.1 has the structure indicated in Theorem 4.12.

In an attempt to complete a characterization of arc-traceable tournaments, we consider whether additional structure imposed on the first and last strong components of $T - z$ is sufficient to guarantee non-arc-traceability.

Lemma 4.13. *Let T be a strong tournament with cut vertex $z \in V(T)$, such that $T - z$ has $k \geq 4$ strong components. Further, let $T_z^{(2)}$ dominate z and let z dominate $T_z^{(k-1)}$ in T. For $x \in T_z^{(1)}$ and $y \in T_z^{(k)}$, xy is part of some hamiltonian path if and only if (i) the vertices of $T_z^{(1)}$ can be partitioned into paths P_1, Q_1 such that P_1 begins at a vertex dominated by z and Q_1 ends at x or (ii) the* *vertices of $T_z^{(k)}$ can be partitioned into paths P_k, Q_k such that Q_k begins at y and P_k ends at a vertex that dominates z.*

Proof. We prove only the sufficiency of condition (i), as condition (ii) is equivalent to condition (i) in the reversal of T. Assume that condition (i) holds. Let P_i be a hamiltonian path of $T_z^{(i)}$ for $2 \leq i \leq k-1$, and let Q_k be any path in $T_z^{(k)}$ from y to a vertex that dominates z. Finally, let P_k be a hamiltonian path of $T_z^{(k)} - V(P_k)$. Then $H = Q_1 Q_k z P_1 P_2 P_2 \ldots P_k$ is a hamiltonian path of T containing the arc xy.

For the converse, assume that T is arc traceable and let H be a hamiltonian path of T containing the arc xy with $x \in T_z^{(1)}$ and $y \in T_z^{(k)}$. First, observe that H contains at most one other arc uv with $u \in T_z^{(1)}$ and $v \notin T_z^{(1)}$, as z must lie between any two such arcs on H. If H does not contain another arc with this property, then the initial vertex of H must be a vertex of $T_z^{(2)}$. In this case, the portion of the path H that lies in $T_z^{(1)}$ is a hamiltonian path of the subtournament $T_z^{(1)}$ that begins at a vertex dominated by z and ends at x. Removing any arc of this subpath yields two paths that satisfy condition (i). So, we may assume that H contains an arc $uv \neq xy$ with $u \in T_z^{(1)}$ and $v \notin T_z^{(1)}$ and in this case, the portion of H that lies in $T_z^{(1)}$ consists of two vertex disjoint paths, P_1 and Q_1 (assume that P_1 precedes Q_1 on H). If xy precedes uv on H, then P_1 ends at x and the vertex immediately preceding Q_1 on H must be z, and so condition (i) is satisfied. If uv precedes xy on H, then there must be an arc $u'v'$ with $u' \notin T_z^{(k)}$ and $v' \in T_z^{(k)}$ such that $u'v'$ precedes xy on H. In the reversal of T, we find that yx precedes $v'u'$ on the reversal of H, and condition (i) is satisfied in $\overline{T}$, and equivalently, condition (ii) is satisfied in T. □

We note that a corollary of this result does give a sufficient condition for non-arc-traceability in strong tournaments.

Corollary 4.14. *Let T be a strong tournament having the structure given by Theorem 4.12. If $|N^+(z) \cap T_z^{(1)}| = 1$ and $|N^-(z) \cap T_z^{(k)}| = 1$, then T is not arc-traceable.*

Proof. Let $N^+(z) \cap T_z^{(1)} = \{x\}$ and $N^-(z) \cap T_z^{(k)} = \{y\}$. Then by Lemma 4.13, xy is non-traceable. □

Note that the converse of this corollary is false, as the tournament in Figure 4.1 is not arc-traceable.

At this point it appears we have characterized arc-traceable tournaments. However, Lemma 4.13 is deceptive. We offer the following corollary to make this clear.

Corollary 4.15. *Let T be a strong tournament having arc xy and the structure given by Theorem 4.12. Let $U = N^+(z) \cap T_z^{(1)}$ and $W = N^-(z) \cap T_z^{(k)}$. Now, if necessary, reverse arcs of $T_z^{(1)}$ to form the tournament R_1 such that x dominates U. Similarly, form the tournament R_k where W dominates y by reversing arcs, if necessary. Then xy is traceable in T if and only if the arc xu is traceable in R_1 for any $u \in U$, or the arc wy is traceable in R_k, for any $w \in W$.*

Proof. If such a traceable arc exists in R_k or R_1, let H be a hamiltonian path containing this arc. Now form paths P and Q that satisfy Lemma 4.13 by deleting the arc xu or wy. Conversely, if there exist two paths that satisfy this Lemma, then these two paths plus an arc xu or wy form a hamiltonian path of R_1 or R_k, respectively. □

It is now clear that our characterization is something of a tautology. A tournament T is not arc-traceable if and only if some other related tournaments R_1 and R_k are not arc-traceable. For this reason, we generally will find it easier to use the incomplete results of Theorem 4.12 to produce sufficient conditions for arc-traceable tournaments (and conversely, necessary conditions for non-arc-traceable tournaments), and will use the additional structure given by Lemma 4.13 less often.

One aspect of Lemma 4.13 that we will use in the next section is that whenever T_z has more than one vertex of minimal in-degree (out-degree), then one of these vertices must dominate (be dominated by) z. We state this as a separate lemma.

Lemma 4.16. *Let T be a non-arc-traceable strong tournament, and let xy be a non-traceable arc of T. If $\delta^0(T) > 1$, then either x is the unique vertex of minimal in-degree in T_z or $\delta^-(T) = \delta^-(T_z)$ for the vertex z in Theorem 4.12. Similarly, either y is the unique vertex of minimal out-degree in T_z or $\delta^+(T) = \delta^+(T_z)$.*

Proof. As $\delta^0(T) > 1$ and $\delta^0(T_z) \geq \delta^0(T) - 1$, we conclude that $\delta^0(T_z) \geq 1$, and the initial and terminal strong components of T_z contain at least three vertices. Let R and S be the initial and terminal strong components of T_z, respectively, and consider R_x and S_y. Vertices of minimal in-degree in T_z must either be x or in the initial strong component of R_x, and by Lemma 4.13, R_x must dominate z (otherwise any hamiltonian path of R_x and the trivial path x imply that the arc xy is traceable). Thus either $\delta^-(T_z) = \delta^-(T)$ or the only vertex of T_z with minimal in-degree is x. In exactly the same way, the terminal strong component of S_y is dominated by z and so either $\delta^+(T_z) = \delta^+(T)$ or the only vertex of T_z with minimal out-degree is y. $\square$

4.3 Applications of the structure results

Although the structure described above falls short of a complete characterization of non-arc-traceable tournaments, it gives a great deal of information about such tournaments. Using this structure, we improve several of the previous sufficient conditions for arc-traceable tournaments as well as improving a result of Volkmann [42] for tournaments with $\delta^0 \geq 2$.

Theorem 4.17. *If T is a strong n-tournament with $\delta^0 \geq 2$ and $d^-(x)+d^+(y) \geq \frac{n}{2} - 2$ for every $xy \in A(T)$, then T is arc-traceable.*

Proof. We prove the contrapositive: if T is not arc-traceable, then for some arc xy of T, $d^-(x) + d^+(y) < \frac{n}{2} - 2$. Assume that T is not arc-traceable, with arc xy on no hamiltonian path of T. Thus, T has the structure given by Theorem 4.12. Let a be the number of vertices in $T_z^{(1)}$ and let b be the number of vertices in $T_z^{(k)}$. Since $T_z^{(2)}$ and $T_z^{(k-1)}$ are distinct and non-empty and $|T_z| = |T| - 1$, $n \geq a+b+3$. Choose vertices u of minimal in-degree and v of minimal out-degree from T_z.

Clearly, $u \in T_z^{(1)}$ and $v \in T_z^{(k)}$, and hence uv is an arc of T_z and so also an arc of T. As $\delta^0(T_z) \geq \delta^0(T) - 1 > 0$, both $T_z^{(1)}$ and $T_z^{(k)}$ contain at least three vertices. Furthermore, if $T_z^{(1)}$ is regular or nearly-regular, then by Lemma 4.16, we can choose u such that $d_T^-(u) = d_{T_z}^-(u) \leq \frac{a-1}{2}$. Otherwise, $d_T^-(u) \leq d_{T_z}^-(u) + 1 \leq \frac{a-3}{2} + 1 = \frac{a-1}{2}$. Similarly, $d_T^+(v) \leq \frac{b-1}{2}$. Thus $d_T^-(u) + d_T^+(v) \leq \frac{a+b-2}{2} \leq \frac{n-5}{2} < \frac{n}{2} - 2$, completing the proof. $\square$

The above result gives Ore-like degree conditions on a tournament that guarantee arc-traceability. As an obvious corollary we state the result with Dirac-like degree conditions.

Corollary 4.18. *If T is a strong n-tournament with $\delta^0 \geq \frac{n}{4} - 1 > 1$, then T is arc-traceable.*

As a consequence of this Dirac-like condition, we state the result in terms of irregularity, similar to Theorem I.

Corollary 4.19. *If T is a strong n-tournament with $\delta^0 > 1$ and $i(T) \leq \frac{n}{2} + 1$, then T is arc-traceable.*

Proof. $i(T) = (n-1) - 2\delta^0$, so $i(T) \leq \frac{n}{2} + 1$ implies that $(n-1) - 2\delta^0 \leq \frac{n}{2} + 1$. Solving for δ^0, we obtain $\delta^0 \geq \frac{n}{4} - 1$. □

The structure of non-arc-traceable tournaments also gives a succinct proof of the following result of Volkmann.

Theorem L (Volkmann [42]). *In a strong tournament T, every arc of T is on some path of length at least $\lceil \frac{n+1}{2} \rceil$.*

Proof. If T is arc-traceable, then every arc of T is on a path of length n and the result is immediate. So we may assume that T is not arc-traceable, and hence for any arc xy that is not on a hamiltonian path, T has the structure indicated in Theorem 4.12.

Let n_i be the number of vertices in the i^{th} component of T_z. Now, let j be the largest integer less than k such that z is dominated (in T) by some vertex in the j^{th} component of T_z. Note, $2 \leq j < k-1$ and some vertex in the $j+1^{\text{st}}$ component of T_z is dominated by z. We can now construct two paths P_1 and P_2, each containing the arc xy with orders $1+n_k+\sum_{i=1}^{j} n_i$ and $1+n_1+\sum_{i=j+1}^{k} n_i$, respectively. Thus, the sum of the orders is at least

$$n_1 + n_k + 2 + \sum_{i=1}^{k} n_i = n_1 + n_k + 2 + (n-1) \geq n+3$$

and one of has order at least $\lceil \frac{n+3}{2} \rceil$, and hence length at least $\lceil \frac{n+1}{2} \rceil$. $\square$

In fact, by combining the above proof and Lemma 4.16 we obtain the following corollary.

Corollary 4.20. *If T is a strong n-tournament with $1 < \delta^0 \leq \frac{n}{4} - 1$, then every arc xy is on a path of length at least $\lceil \frac{n+1}{2} \rceil + 2\delta^0$.*

We present one additional degree condition based on Theorem 4.12, a local condition that is sufficient to guarantee that a given arc is traceable.

Theorem 4.21. *Let T be a strong tournament with $\delta^0 > 1$ and let $xy \in A(T)$. If $d^+(y) > d^+(x) - 4$, then xy is traceable.*

Proof. Assume that xy is non-traceable. Then T has the structure given by Theorem 4.12. Furthermore, as $\delta^0 > 1$, we can choose u_1 from $T_z^{(1)}$ such that x dominates u_1, and u_2 from $T_z^{(k)}$ such that u_2 dominates y. Next, we can choose u_3 and u_4 from $T_z^{(2)}$ and $T_z^{(k-1)}$, respectively. From the structure of T_z, x dominates u_i and u_i dominates y for each i, $1 \leq i \leq 4$. Additionally, x dominates y, so $\{u_1, u_2, u_3, u_4, y\} \subseteq N^+(x) \setminus N^+(y)$. Finally, $N^+(y) \setminus N^+(x) \subseteq \{z\}$, and so we have

$$d^+(y) = d^+(x) + |N^+(y) \setminus N^+(x)| - |N^+(x) \setminus N^+(y)| \leq d^+(x) - 4.$$

□

Next, we give two results that relate arc-traceability to kings in tournaments. A *king* in a tournament is a vertex that reaches all other vertices via a directed path of length at most two. Tournaments in which every vertex is a king were studied in [35] and [8]. The first result, for all tournaments, is sufficient to show that all-kings tournaments are arc-traceable.

Theorem 4.22. *If T is an n-tournament with n or $n-1$ kings, then T is arc-traceable.*

Proof. Assume that T is not arc-traceable, and hence has the structure given in Theorem 4.12. Choose v_1 and v_2 from $T_z^{(k-1)}$ and $T_z^{(k)}$, respectively and observe that for any vertex $u \in T_z^{(2)}$, the shortest path from v_i to u has length at least three. Thus neither v_1 nor v_2 are kings and so T has at most $n-2$ kings. □

When $\delta^0 > 1$, similar techniques give a slightly improved result.

Theorem 4.23. *If T is a non-arc-traceable strong n-tournament with $\delta^0 > 1$, then T has at most $n-3-2\delta^+$ kings.*

Proof. Assume that T is non-arc-traceable, and let xy be a non-traceable arc of T. Then T has the structure given by Theorem 4.12, and since $\delta^- > 1$, $T_z^{(1)}$ has order at least three. As a consequence of Lemma 4.13, there must be some vertex $v \in T_1^{(1)}$ with $v \to z$. Now, for any vertex $w \in T_z^{(j)}$ for $j > 1$, the shortest path from w to v has length at least three. Hence we conclude that the set of kings of T is contained in $V(T_z^{(1)}) \cup \{z\}$. We count the vertices not in this set and find at least one vertex each in $T_z^{(2)}$ and $T_z^{(k-1)}$ and at least

$2\delta^+(T_z)+1$ vertices in $T_z^{(k)}$. Thus, there are at most $n-2-(2\delta^+(T_z)+1)$ kings of T, and the proof is complete if $\delta^+(T_z) = \delta^+(T)$. Otherwise, since $\delta^0 > 1$, Lemma 4.16 implies that $T_z^{(k)}$ is not regular or nearly regular, and so must have at least $2\delta^+(T_z)+3$ vertices. In this case, we again have at most $n-2-(2\delta^+(T_z)+3) = n-2-(2(\delta^+(T)-1)+3) = n-3-\delta^+(T)$ kings in T. □

4.4 Extremal results related to arc-traceable tournaments

In this section, we use the structure of non-arc-traceable strong tournaments to produce some extremal results. First, we fix n and consider the maximum number of non-traceable arcs in a strong n-tournament. Later, we consider tournaments with m arc-disjoint paths between any two vertices, and we determine the minimal n such that some m-arc-strong n-tournament is not arc-traceable.

In Chapter 3, we constructed an upset n-tournament with $\frac{n^2-4n+3}{8}$ non-traceable arcs for any odd n, and we proved that this tournament was maximal with respect to the number of non-traceable arcs. We will denote this tournament $T_{\max}$, and recall that this tournament is obtained from the transitive tournament on the set $V = \{v_0, \dots v_{n-1}\}$ where $d^+(v_i) = i$ by reversing the arcs $v_i v_{i+2}$ for each even $i < n-1$. We now prove that any strong n-tournament T has at most this number of non-traceable arcs with equality if and only if T is isomorphic to $T_{\max}$.

Lemma 4.24. *Let T be a strong tournament having a cut-vertex z. If $X = \{x : T_z^{(1)}$ can not be covered by paths P and Q such that P begins at a vertex dominated by z and Q ends at $x\}$, then $|X| \le \frac{a+1}{2}$, where $a = |T_z^{(1)}|$. Similarly, $|Y| \le \frac{b+1}{2}$ for the analogous set Y, where $b = |T_z^{(k)}|$.*

Proof. Assume that $X = \{x_0, x_1, \ldots, x_m\}$. If $|N^+(z) \cap X| \geq 1$, then assume that z dominates x_0. Let P_i be the longest path not containing x_i that begins at a vertex dominated by z. As z dominates x_0 or some some vertex $z^+ \notin X$, P_i is a path containing at least one vertex for each i, $1 \leq i \leq m$. Let $S_i = V(T_z^{(1)}) \setminus V(P_i)$ for $1 \leq i \leq m$. We claim first that $S_i \setminus \{x_i\}$ dominates $V(P_i)$. Assume otherwise, and let v be the last vertex along P_i such that v dominates w for some $w \in S_i \setminus \{x_i\}$. If v is the terminal vertex of P_i, then $P_i w$ is a longer path than P_i beginning at a vertex dominated by z. Otherwise, by the minimality of v, we can replace the arc vv^+ of P_i with the 2-path vwv^+ and again obtain a path that begins at vertex dominated by z that is longer than P_i. Note, since $T_z^{(1)}$ is strong, S_i must be reachable from $V(P_i)$, and thus some vertex of $V(P_i)$ must dominate x_i. Choose such a vertex and call it v_i.

Now, let Q_i be the longest path of $T[S_i]$ that ends at x_i, and let $U_i = S_i \setminus V(Q_i)$. By the definition of X, $U_i \neq \emptyset$ for $1 \leq i \leq m$ and $U_i \subset S_i \setminus \{x_i\}$, so U_i dominates $V(P_i)$. By a similar argument used above, we also note that U_i is dominated by each vertex of $V(Q_i)$. Thus, we conclude that $V(Q_i) \setminus \{x_i\}$ dominates both U_i and $V(P_i)$.

Additionally, observe that the terminal strong component of $T[U_i]$ contains no vertex of X and hence $U_i \setminus X \neq \emptyset$. To see this, let H be any hamiltonian path of $T[U_i]$ ending at some vertex of X, and construct the paths $Q_i H$ and P_i, which partition the vertices of $T_z^{(1)}$. The initial vertex of P_i begins at a vertex dominated by z, and so by definition the terminal vertex of $Q_i H$ is not in X. Since we can choose H so that it ends at any vertex in the terminal strong component of $T[U_i]$, this terminal strong component is disjoint from X.

Finally, we claim that $U_i \cap U_j = \emptyset$ for all $i \neq j$. Assume otherwise, and choose $i \neq j$ with $u \in U_i \cap U_j$. Without loss of generality assume that x_i dominates x_j. Since $u \in U_i$, and U_i dominates $V(P_i)$, u dominates v_i. Similarly, as $V(Q_i)$ and $V(Q_j)$ dominate $U_i \cap U_j$, x_i and x_j both dominate u. Since v_i does not dominate u and $V(Q_j)$ dominates u, $v_i \notin V(Q_j)$, and hence $v_i \in V(P_j) \cup U_j$. Also, since x_i dominates x_j by assumption, and $V(Q_j)$ dominates U_j, $x_i \notin U_j$. Further, x_i dominates u and U_j dominates $V(P_j)$ so $x_i \notin V(P_j)$. Thus, $x_i \in V(Q_j) \setminus \{x_j\}$ and so x_i dominates both U_j and $V(P_j)$. Since $v_i \in V(P_j) \cup U_j$, this requires that x_i dominates v_i, contradicting the choice of v_i as a vertex of P_i that dominates x_i.

The above arguments show that $U_i \setminus X \neq \emptyset$ for $1 \leq i \leq m$ and $U_i \cap U_j = \emptyset$ for $i \neq j$, which establishes that

$$\left| \left(\bigcup_{i=1}^{m} U_i \right) \setminus X \right| = \sum_{i=1}^{m} |U_i \setminus X| \geq m = |X| - 1.$$

Thus, we have $a \geq |X| + m = 2m + 1 = 2|X| - 1$ and so $|X| \leq \frac{a+1}{2}$.

The bound for the set Y is obtained using an identical argument in the reversal of T. $\square$

Corollary 4.25. *Let T be a strong tournament having the structure given by Theorem 4.12. The number of non-traceable arcs from $T_z^{(1)}$ to $T_z^{(k)}$ is at most $\frac{(ab+a+b+1)}{4}$ where $a = |T_z^{(1)}|$ and $b = |T_z^{(k)}|$.*

Proof. Let B be the set of arcs we wish to count. Define the sets X and Y as in Theorem . By Lemma 4.13, an arc $a = uv$ is in the set B if and only if $u \in X$ and $v \in Y$ and so $|B| = |X||Y|$ and by Lemma 4.24, $|X||Y| = \frac{(a+1)(b+1)}{4}$. $\square$

Theorem 4.26. *If T is a strong n-tournament, then T has at most $\frac{n^2-4n+3}{8}$ non-traceable arcs, with equality if and only if T is isomorphic to T_{max}.*

Proof. For $n = 3$ and $n = 4$, there is a unique strong n-tournament, and in both cases this tournament is arc-traceable and so has at most $0 = \lfloor \frac{4^2-4(4)+3}{8} \rfloor$ non-traceable arcs, and this tournament is isomorphic to $T_{\max}$.

For $n > 4$, assume that T is non-arc-traceable. Let xy be a non-traceable arc of T, and let T have the structure given by Theorem 4.12. Let $A = T_z^{(1)}$ and $B = T_z^{(k)}$, with $a = |A|$ and $b = |B|$, and choose $u \in T_z^{(2)}$ and $w \in T_z^{(k-1)}$. If $a = x'y'$ is a non-traceable arc of T, neither $T - x'$ nor $T - y'$ are strong, and hence x' and y' must both be in the set $A \cup B \cup \{z\}$. Thus, a is either an arc of $T[A \cup \{u, z\}]$, an arc of $T[B \cup \{w, z\}]$, or an arc from A to B.

We claim that every arc of $T[A \cup \{u, z\}]$ or $T[B \cup \{w, z\}]$ that is non-traceable in T is also non-traceable in this subtournament. Assume that some arc a is traceable in $T[A \cup \{u, z\}]$, and let P be a hamiltonian path of this tournament containing the arc a. If P ends at a vertex other than z, then for any hamiltonian path Q of the vertices not on P, PQ is a hamiltonian path of T containing a. On the other hand, u has out-degree one in this subtournament, so if H ends at z it must end with the arc uz. If a is not the arc uz, then we can remove this arc from the end of P to obtain the path P'. Then for any hamiltonian path of the remaining vertices Q such that Q ends at a vertex dominating z, $P'Qz$ is a hamiltonian path of T containing a. Finally, if $a = uz$, then we can let P' be any hamiltonian path of $T_z^{(1)}$ beginning at a vertex dominated by z and if Q is any hamiltonian path of the remaining vertices of $T_z - u$, $aP'Q$ is a hamiltonian path of T. An identical argument in the reversal of T establishes the corresponding result for $T[B \cup \{w, z\}]$.

As both of the subtournaments $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ are strong, we can apply the induction hypothesis, and hence $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ have at most $\frac{(a+2)^2-4(a+2)+3}{8}$ and $\frac{(b+2)^2-4(b+2)+3}{8}$ non-traceable arcs, with equality if and only if each of these subtournaments are isomorphic to $T_{\max}$. Summing these two values, we obtain $\frac{a^2+b^2-2}{8}$.

Finally, by Corollary 4.25, there are at most $\frac{(a+1)(b+1)}{4}$ non-traceable arcs from A to B. Combining and observing that $a + b \leq n - 3$, the number of non-traceable arcs in T is at most

$$\begin{aligned}
\frac{a^2+b^2-2}{8} + \frac{ab+a+b+1}{4} &= \frac{(a^2+2ab+b^2)+2(a+b)}{8} \\
&= \frac{(a+b)^2+2(a+b)}{8} \\
&\leq \frac{(n-3)^2+2(n-3)}{8} \\
&\leq \frac{n^2-6n+9+2n-6}{8} \\
&\leq \frac{n^2-4n+3}{8}
\end{aligned}$$

The proof is complete by noting that in the above equation, equality is established if and only if $n = a + b + 3$ and the subtournaments $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ are both upset tournaments isomorphic to the appropriate size $T_{\max}$. From this it follows directly that T is also an upset tournament and isomorphic to $T_{\max}$. □

As noted earlier, a consequence of Theorem 4.7 implies that for any non-traceable arc xy in a strong tournament T, there exists a y, x separating set of

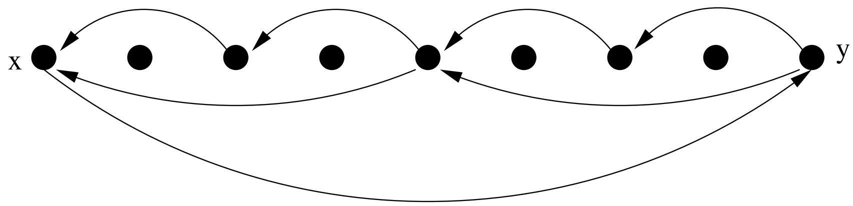


Figure 4.2: A tournament with non-traceable arc xy and two arc-disjoint paths from y to x.

vertices of order 1 (the vertex z in the theorem). We now show that a similar result for a separating set of arcs is impossible. In particular, we show that for any $m > 0$ there exists a strong tournament T that has a non-traceable arc xy with m arc-disjoint paths from y to x. As an example, in the tournament in Figure 4.2, xy is not on any hamiltonian path, and there exist 2 arc-disjoint paths from y to x.

For convenience, throughout the remainder of this chapter we will let $n = 2^{m+1}$. For an arbitrary m, we construct a strong tournament by reversing the arcs of a set of m arc-disjoint paths in a transitive tournament of order $n+1$. Let T be a transitive $(n+1)$-tournament with vertices labeled $v_0, \ldots, v_n$ such that $d^+(v_i) = i$. Now consider the paths

$$P_i = v_n v_{(n-2^i)} v_{(n-2\cdot 2^i)} v_{(n-3\cdot 2^i)} \ldots, v_0, \text{ for } 1 \leq i \leq m.$$

We reverse the arcs in each of these paths to obtain the tournament $T_{[m]}$, and will refer to the reversed path P_i as the i^{th} upset path and denote it U_i. Note, $v_{\frac{n}{2}}$ is on each U_i, and thus for any $i < \frac{n}{2} < j$, $v_{\frac{n}{2}}$ separates v_i and v_j. Also, observe that we can view the construction of $T_{[m+1]}$ recursively; take two copies of $T_{[m]}$ sharing a single vertex (the terminal endpoint of all the upset paths from one copy, and the initial endpoint of the upset paths in the other copy) and reverse the 2-path P_{m+1}.

Lemma 4.27. *The arc $v_n v_0$ is not on any hamiltonian path of the tournament $T_{[m]}$.*

Proof. For $m = 1$, $T_{[1]}$ is an upset tournament on 5 vertices, and it is easy to verify that $v_5 v_0$ is not on any hamiltonian path of $T_{[1]}$. Now, assume the result for m and consider $T_{[m+1]}$. Consider P, a path in $T_{[m+1]}$ containing the arc $v_{2n} v_0$ of maximal length, and as $T_{[m+1]}$ is isomorphic to its reversal, without loss of generality assume that v_n precedes v_{2n} on P. Let v_t be the terminal vertex of P and v_i the first vertex of P with index $i \leq n$. Since v_n separates v_a from v_b for each $a < n < b$, every vertex between v_i and v_n must have index $j < n$. Similarly, every vertex between v_n and v_{2n} must have index $j > n$ and every vertex following v_0 must have index $j < n$. Thus, all the vertices of P with index $j \leq n$ are contained in the subpaths $v_i \dots v_n$ and $v_0 \dots v_t$. We now consider the combined sequence of vertices $Q = v_i, \dots, v_n, v_0, \dots, v_t$. Each of the vertices in this sequence has index $j \leq n$, and every pair of consecutive vertices other than $v_n v_0$ are joined by an arc of $T_{[m+1]}$. Since $v_n v_0 \in A(T_{[m]})$, we can use the recursive perspective described earlier and think of this sequence as a path of $T_{[m]}$. Since Q contains the arc $v_n v_0$, by the induction hypothesis, this can not be a hamiltonian path of $T_{[m]}$, and so there is a vertex v_a, $a < n$, that is not in the sequence Q. By the minimality of i, v_a can not precede v_i, and v_n separates v_a from v_{2n}, so v_a can not lie between v_n and v_{2n} on P. Hence v_a is not on the path P. Consequently, the longest path containing the arc $v_{2n} v_0$ is not hamiltonian, and $v_{2n} v_0$ is non-traceable. □

Thus for any m, we can construct a tournament with $n+1$ vertices with a non-traceable arc xy such that there exist m arc-disjoint paths from y to x. Next we show that this is the minimal number of vertices among strong tournaments with this property.

Theorem 4.28. *If T is a strong k-tournament containing a non-traceable arc xy such that there exist m arc-disjoint paths from y to x, then $k \geq n+1 = 2^{m+1}+1$.*

Proof. Again, the proof is by induction. For $m = 1$, the result is obtained by observing that the unique strong tournaments on 3 and 4 vertices are arc-traceable. Next, assume the result for m and consider the smallest strong k-tournament T, with non-traceable arc xy and $m+1$ arc-disjoint paths from y to x. Assume that $k < 2n+1 = 2^{m+2}+1$.

As xy is non-traceable, T must have the structure given by Theorem 4.12. Furthermore, the minimality of T implies that T_z has exactly four strong components, and that the second and third components both consist of a single vertex. Let X be the set of vertices in the first and second strong components of T_z, and $Y = V(T_z) \setminus X$. Clearly, either $|X| < n$ or $|Y| < n$. Without loss of generality, assume $|X| < n$ and consider the tournament $T[X \cup \{z\}]$. If $zx \in A(T)$, then reverse this arc to form the tournament T'. Otherwise, simply let $T' = T[X \cup \{z\}]$. Since z is on every path from y to x, and there are $m+1$ arc-disjoint paths from y to x, there also exist $m+1$ arc disjoint paths from z to x in T and at most one of these contains the arc zx (if it is an arc of T). Thus, there are at least m arc-disjoint paths from z to x in T'. Clearly, T' is strong and has fewer than $n+1$ vertices, and so by the induction hypothesis, xz is on some hamiltonian path of this tournament. Let P be such a hamiltonian path, and split P into two smaller paths P_1, consisting of all the vertices up to and including x and P_2, consisting of all the vertices of P that follow z. The structure of T' guarantees that both P_1 and P_2 are paths of order at least 1. The only vertex of T' on neither P_1 or P_2 is z, so each vertex of X is on either P_1 or P_2. Next, the structure of T requires that $T[Y \cup \{z\}]$ must be strong, so let C be a hamiltonian cycle of this tournament and let Q_1 be the subpath of C from y to z, inclusive, and Q_2 the subpath of C from the vertex immediately succeeding z to the vertex immediately preceding y. We allow for the possibility that Q_2 may have order 0. Thus, every vertex of $Y \cup \{z\}$ is on either Q_1 or Q_2. We now construct $H = P_1Q_1P_2Q_2$, and we claim that H is a hamiltonian path of T. First, the terminal vertex of P_1 is x and the initial vertex of Q_1 is y, and $xy \in A(T)$ by assumption. Next, the terminal vertex of Q_1 is z, and the initial vertex of P_2 is the vertex immediately following z on P. Lastly, the terminal vertex of P_2 is a vertex of X, while the initial vertex of Q_2 (if any) is a vertex of Y, and X dominates Y. So, H is indeed a path of T. That H is hamiltonian is immediate; it includes all of X and Y as well as the vertex z. Finally, H includes the arc xy. But xy is non-traceable, contradicting our assumption that $k < 2n + 1$. □

The previous results apply to a particular arc xy and as a result we make no claim about the number of arc-disjoint paths between every two vertices. In fact, for each m, $T_{[m]}$ contains a vertex of in-degree 1 ($v_{(n-1)}$) as well as a vertex of out-degree 1 (v_1), so we have yet to produce even a 2-arc-strong tournament that is not arc-traceable. We now seek to construct such an m-arc-strong non-arc-traceable tournament. Doing so requires only a minor variation on the construction of $T_{[m]}$.

Specifically, let $T'_{[m]}$ be a tournament on $n + 4m - 3$ vertices obtained from $T_{[m]}$ by removing the vertices v_1 and v_{n-1} and adding sets U_1 and U_{n-1} each of order $2m - 1$. Orient the edges incident with vertices of $U_1 \cup U_{n-1}$ such that $T[U_1]$ and $T[U_{n-1}]$ are regular tournaments of degree $m-1$, and U_{n-1} dominates every other vertex of $T'_{[m]}$ except v_n, and U_1 is dominated by every remaining vertex of $T'_{[m]}$ except v_0.

Lemma 4.29. *$T'_{[m]}$ is m-arc-strong.*

Proof. It suffices to show that v_0 both reaches and is reached by every other vertex of $T'_{[m]}$ using m arc-disjoint walks. The result then follows by a variant of Theorem E.

To show that each vertex reaches v_0 in m different ways, first consider any vertex $u \in U_1$ and let $N^+(u) \cap U_1 = \{u_1, \ldots, u_{m-1}\}$. Clearly, $W_i = uu_iv_0$ for $1 \leq i \leq m-1$ and $W_m = uv_0$ are m arc-disjoint walks from u to v_0. Additionally, for any vertex $v \notin U_1$, $W_i = vu_iv_0$ for $1 \leq i \leq m-1$ and $W_m = vuv_0$ are arc-disjoint walks from v to v_0.

To show that v_0 reaches every other vertex by m arc-disjoint walks, we use the set of m upset paths $U_1, \ldots U_m$. Clearly, $U_1, \ldots, U_m$ are arc-disjoint paths from v_0 to v_n. To show that v_0 reaches every vertex of $w \in U_{n-1}$ by m arc-disjoint walks, let $N^-(w) \cap U_{n-1} = \{w_1, \ldots w_{m-1}\}$ and let $W_i = U_iw_iw$ for $1 \leq i \leq m-1$ and $W_m = U_mw$. As above, for $v \notin U_{n-1} \cup \{v_n\}$, let $W_i = U_iw_iv$ for $1 \leq i \leq m-1$ and $W_m = U_mwv$. □

Lemma 4.30. *The arc v_nv_0 of $T'_{[m]}$ is non-traceable.*

Proof. Assume the result is false, so there is some hamiltonian path H of $T'_{[m]}$ that contains the arc v_nv_0. Since $N^-(u) \setminus U_{n-1} = \{v_n\}$ for any $u \in U_{n-1}$, any path ending at a vertex of U_{n-1} is either a path of $T[U_{n-1}]$, or includes an arc v_nw for some $w \in U_{n-1}$. Since H can not contain any of these arcs, every subpath of H ending at a vertex of U_{n-1} is a path of $T[U_{n-1}]$ and hence the first $2m-1$ vertices of H are precisely the vertices in the set U_{n-1}. Similarly, every subpath of H beginning at a vertex of U_1 is a path of $T[U_1]$ and hence the last $2m-1$ vertices of H are precisely the vertices in U_1. Now, the subpath Q of H beginning with first vertex of H not in U_{n-1} and ending at the last vertex of H not in U_1 is also a path of $T_{[m]}$ containing v_nv_0, and this path has order $n+4m-3-(4m-2) = n-1$. Furthermore, this path cannot begin at the vertex v_n because this vertex is in $N^+(u)$ for some $u \in U_{n-1}$. Similarly, the last vertex of Q is in $N^-(w)$ for some $w \in U_1$, so Q does not end at the vertex v_0. Thus, $v_{n-1}Qv_1$ is a path of $T_{[m]}$ with order $n+1$, and hence $v_{n-1}Qv_1$ is hamiltonian. But this path contains v_nv_0, contradicting Theorem 4.7. Thus, v_nv_0 is non-traceable in $T'_{[m]}$. $\square$

Finally, we conclude this chapter with a proof that $T'_{[m]}$ has the fewest vertices among all non-arc-traceable m-arc-strong tournaments.

Theorem 4.31. *If T is a non-arc-traceable m-arc-strong k-tournament, then $k \geq n+4m-3$.*

Proof. For $m=1$, the result is immediate by observing that all strong tournaments are 1 arc-strong and that $2^{1+1}-4(1)-3=5$ is the size of the smallest non-arc-traceable strong tournament. So we can assume that $m \geq 2$. Let xy be a non-traceable arc of T. Define S_x as the initial strong component of $T-x$ and S_y as the terminal strong component of $T-y$. If $y \in S_x$, there would exist a hamiltonian path of T beginning with the arc xy. So $y \notin S_x$ and similarly, $x \notin S_y$. As T is m arc-strong, $\delta^0_T \geq m$ and $\delta^0_{T-v} \geq m-1$ for any $v \in V(T)$. Thus, $\delta^-_{T-x} \geq m-1$ and $\delta^+_{T-y} \geq m-1$ and consequently both $|S_x| \geq 2m-1$ and $|S_y| \geq 2m-1$.

Next, we claim that $S_x \cap S_y = \emptyset$. Assume otherwise, and choose $z \in S_x \cap S_y$. Since $z \in S_x$, it reaches every vertex of $T-x$, and since $y \notin S_x$, it reaches every vertex of $(T-x)-y$. So z is in the terminal strong component of $T-y$ but the initial strong component of $T-y-x$. As a result, the initial strong component of $T-y$ must be $\{x\}$. This requires that $\delta^-_{T-y} = 0 < m-1$, a contradiction.

Lastly, observe that no path from y to x can use any vertex of $S_x \cup S_y$, as every path from y to S_x must contain the vertex x, and dually every path from S_y to x must contain y. Thus we can form T' by replacing the entire set S_x with a single vertex u_x and replacing the entire set S_y with a single vertex u_y without disturbing any path from y to x. Thus, there remain m arc-disjoint paths from y to x in T'. Furthermore, if we let x dominate u_x and let u_y dominate y, then T' is strong. By a similar argument to the one used in Lemma 4.30, any hamiltonian path of T' containing the arc xy can be extended to a hamiltonian path of T containing this arc. As xy is not on any hamiltonian path of T by assumption, it is therefore not on any hamiltonian path of T'. By Lemma 4.28, T' has at least $n+1$ vertices and so T has at least $(n+1)-2+2(2m-1) = n+4m-3$ vertices. □

5. Related problems

In this concluding chapter, we consider arc-traceability in other classes of digraphs related to tournaments, as well as various other ways to generalize arc-traceability. We will introduce new definitions and notation as needed. The main result of this chapter will be a characterization of arc-traceability in the class of strong round-decomposable locally semicomplete digraphs. These results are intended to suggest new directions for the investigation of arc-traceable digraphs and to propose the study of properties similar to arc-traceability that bear further study.

5.1 Semicomplete digraphs

A *semicomplete* digraph is a strict digraph D, such that for every pair of distinct vertices $x, y \in V(D)$, either $xy \in A(D)$ or $yx \in A(D)$ or both. In other words, a semicomplete digraph is any strict digraph that contains a tournament as a spanning subdigraph. As such, we would like to be able to generalize the techniques and results of Chapter 4 in a natural way to obtain results for this class of digraphs.

For example, we note that just like tournaments, the condensation D^* of a semicomplete digraph D is a transitive tournament. Thus, just as for tournaments we have the following observation.

Proposition 5.1. *Let D be a semicomplete digraph. If D is arc-traceable, then D is strong or has two strong components D_1 and D_2, each of which are arc-traceable semicomplete digraphs.*

Next, we will show that every 2-connected semicomplete digraph is arc-traceable. The argument used in Chapter 4 for tournaments only uses the fact that strong tournaments are hamiltonian. So our next observation follows by observing that a strong semicomplete digraph is hamiltonian due to Theorem D.

Proposition 5.2. *Every 2-connected semicomplete digraph is arc-traceable.*

Continuing in this way, a careful reading of Lemma 4.3 and Theorem 4.4 shows that these results hold for semicomplete digraphs as well as tournaments. Thus we obtain the same sufficient condition given by Corollary 4.5.

Corollary 5.3. *If D is a strong semicomplete digraph, and every arc of D is on some cycle of length $l > \frac{n+1}{2}$, then D is arc-traceable.*

Up to this point, we have obtained exactly the same results for semicomplete digraphs that we did for tournaments. However, an analog of Theorem 4.7 is not quite as immediate. If an arc xy of a semicomplete digraph D is part of a cycle of length two, then it makes no sense to speak of a y, x-separating set and we are not able to use Menger's Theorem. Apart from this difficulty, however, the proof of Theorem 4.7 remains valid even when other 2-cycles remain in the digraph. This yields a result that is only slightly modified.

Theorem 5.4. *Let D be a strong semicomplete digraph and let $xy \in A(D)$ be such that there exist at least two (internally) disjoint paths of length at least two from y to x in D. Then there exists a hamiltonian path of D containing the arc xy.*

As a result of this modification, the structure of a non-arc-traceable strong semicomplete digraph is similar but not identical to the structure for tournaments. To make the analog of Theorem 4.12 as simple as possible, we define the subdigraph $D_{(uv)}$ of D by $V(D_{(uv)}) = V(D)$ and $A(D_{(uv)}) = A(D) \setminus \{uv\}$. So, in the case where $uv \notin A(D)$, $D_{(uv)} = D$. Certainly, no path including the arc xy can also include the arc yx. Thus, we observe that xy is traceable in D if and only if xy is traceable in $D_{(yx)}$. We now give the main structure theorem for semicomplete digraphs, which is an amalgam of the semicomplete digraph analogs of Theorem 4.12 and Corollary 2.2.

Theorem 5.5. *If D is a strong semicomplete digraph and $xy \in A(D)$ is not on a hamiltonian path, then either D_{yx} has three or more strong components, or there exists a vertex $z \in V(D_{(yx)})$ such that:*

- *$D_{(yz)} - z$ has $k \geq 4$ strong components.*
- *x is in the initial strong component, and y is in the terminal strong component of $D_{(yx)} - z$.*
- *z is dominated by the 2^{nd} strong component and dominates the $(k-1)^{st}$ strong component of $D_{(yx)} - z$*

To illustrate Theorem 5.5, we give two non-arc-traceable strong semicomplete digraphs in Figure 5.1. The digraph on the left has non-traceable arc xy, and the digraph obtained by removing the corresponding arc yx has three strong components. The right hand digraph has the structure similar to non-arc-traceable strong tournaments, but with the additional arc yx. In both cases, additional 2-cycles may be present within the components.

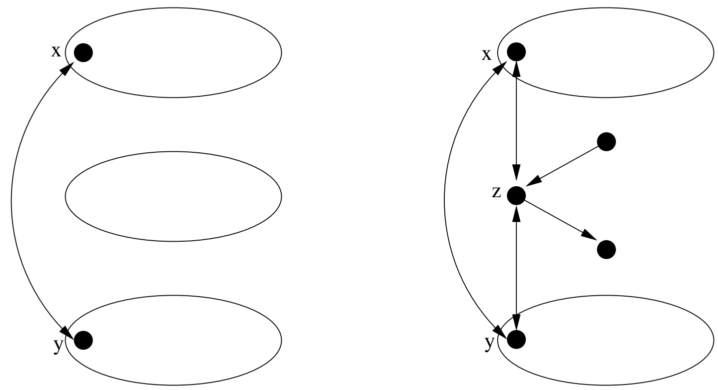


Figure 5.1: Two non-arc-traceable strong semicomplete digraphs.

Proceeding in this fashion, we can use the slightly modified structure given above to yield results for semicomplete digraphs that closely match the results for tournaments. We illustrate this by giving degree constraints sufficient to guarantee arc-traceability. Observe that these conditions differ significantly from the conditions for tournaments because in the strong components of a semicomplete digraph we allow cycles of length two.

Theorem 5.6. *If D is a strong semicomplete digraph on n vertices with $\delta^0 \geq 2$, and for every $xy \in A(D)$, $d^-(x) + d^+(y) \geq n - 2$, then D is arc-traceable.*

Proof. Assume D is not arc-traceable. Then by Theorem 5.5, either $D_{(yx)}$ has three or more strong components or we can find a vertex z as described in the theorem.

If $D_{(yx)}$ has three or more strong components, then let a and b be the number of vertices in the initial and terminal strong components, respectively. Note, $n \geq a+b+1$, and since $\delta^0 > 1$, both a and b are at least two. This implies that we can choose vertices u and v in these components with $d_D^-(u) \leq a-1$ and $d_D^+(v) \leq b-1$. It must be the case that $u \to v$ and hence we have an arc of D with $d^-(u) + d^+(v) \leq a+b-2 = n-3$.

If instead, $D_{(yx)}$ has a vertex z as described in Theorem 5.5, let the number of vertices in the first strong component of $D_{(yx)} - z$ be a and let b be the number of vertices in the last strong component. This requires that $n \geq a+b+3$, and both a vertex u with $d^-_{D_{(yx)}}(u) \leq a-1$ and a vertex v with $d^+_{D_{(yx)}}(v) \leq b-1$. Again, it is clear that $u \to v$ in D and for this arc, $d_D^-(u) + d_D^+(v) \leq \big(d^-_{D_{(yx)}}(u) + 1\big) + \big(d^-_{D_{(yx)}}(u) + 1\big) = a+b = n-3$. □

5.2 Round digraphs

Another generalization of tournaments in which arc-traceability can be considered is the class of digraphs known as *local tournaments.* A digraph is a local tournament if the subdigraphs induced on $N^+(v)$ and $N^-(v)$ are tournaments for every vertex v. Originally introduced as *locally semicomplete* digraphs in [5], the structure of local tournaments was investigated by Huang in [19].

Our analysis of arc-traceable locally semicomplete digraphs begins with a special case of local semicomplete digraphs called *round digraphs.* A digraph D is said to be round if there is an ordering of $V(D)$ such that $N^-(v_i) = \{v_{i-j} \mid 1 \leq j \leq d^-(v_i)\}$ and $N^+(v_i) = \{v_{i+j} \mid 1 \leq j \leq d^+(v_i)\}$ for each $v_i \in V(D)$ (addition of subscripts is modulo n). The ordering is called a *round labeling* or *round enumeration* of the digraph D. We give a few observations about round digraphs in general.

Theorem M (Huang [20]). *A round digraph is locally semicomplete.*

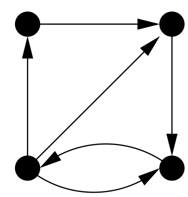


Figure 5.2: A round local tournament with cycles of length two.

Proof. Let D be a round digraph with round labeling $V(D) = \{v_0, \dots v_{n-1}\}$. If $d^+(v_i) \leq 1$, then the out-neighborhood of v_i is trivially a semicomplete digraph. So for any i such that $d^+(v_i) > 1$, choose v_j and v_k from $N^+(v_i)$. Without loss of generality, assume that $i < j < k$. Now since $v_i \in N^-(v_k)$, the round labeling implies that $v_j \in N^-(v_k)$. Thus, every pair of distinct vertices in $N^+(v_i)$ are adjacent. Similarly, every pair of distinct vertices in $N^-(v_i)$ are adjacent, D is locally semicomplete. □

Since a semicomplete digraph with no 2-cycles is a tournament, we get a sufficient condition for a round digraph to be a local tournament.

Proposition 5.7. *A round digraph with no cycles of length two is a local tournament.*

The converse of this statement is false, however, as shown by the round digraph in Figure 5.2 which is a local tournament despite having a cycle of length two.

It is easy to see from the round labeling that any round digraph with $\delta^+ \geq 1$ is hamiltonian, since $v_i \to v_{i+1}$ for each i, $0 \leq i \leq n-1$. A similar argument gives a sufficient condition for a round digraph to be arc-traceable.

Theorem 5.8. *A round digraph D with $\delta^+ \geq 2$ is arc-traceable.*

Proof. Let D be a round digraph with $\delta^+ \geq 2$, and let $V(D) = \{v_0, \ldots, v_{n-1}\}$ be a round labeling of D. Now choose any arc $a = v_iv_j$ of D. If $j = i+1$, then a is on a hamiltonian cycle, so we may assume that $j \neq i+1$. Consider the vertex v_{j-1}. Since $d^+(v_{j-1}) > 2$, the round labeling requires that $v_{j-1} \to v_{j+1}$ and we can construct the hamiltonian path $v_{i+1}v_{i+2} \cdots v_{j-1}v_{j+1}v_{j+2} \ldots v_iv_j$. □

Before returning to the case of $\delta^+ = 1$, we consider non-strong round digraphs. Clearly, these are precisely the round digraphs with $\delta^+ = 0$. We show that such digraphs are acyclic.

Lemma 5.9. *If D is a round digraph that is not strongly connected, then D is acyclic.*

Proof. Let D be a round digraph with the usual round labeling, and let C be a directed cycle in this digraph. Then some arc of C must be of the form v_jv_i with $i < j$. The round labeling forces $N^+(v_j) \subseteq \{v_{j+1}, \ldots, v_{n-1}, v_0, \ldots, v_i\}$ and so we have $v_jv_0 \in A(D)$. Again, we use the round labeling, which implies that $N^-(v_0) \subseteq \{v_j, \ldots, v_{n-1}\}$. Using the round labeling a third time, we find that $N^-(v_i) \subseteq \{v_j, \ldots, v_{n-1}, v_0, \ldots, v_{i-1}\}$. This implies that $d^+(v_k) \geq 1$ for $0 \leq k < i$ and $j \leq k \leq n-1$ and since v_i is on the cycle C, $d^+(v_i) \geq 1$ as well.

Now certainly, for any k with $i < k < j$, C contains some arc v_av_b with $a < k < b$. Thus, using the round labeling once again, $v_k \in N^-(v_b)$ and so $d^+(v_k) \geq 1$ for each $i < k < j$ and hence we have $\delta^+ \geq 1$ and D is strong. □

As a consequence of the above lemma, we observe that a round digraph that is not strongly connected has at most one hamiltonian path.

Proposition 5.10. *An acyclic digraph has at most one hamiltonian path*

Proof. Let $H_1 = v_1 v_2 \cdots v_n$ be a hamiltonian path of a directed graph D. Then if D contains any other hamiltonian path H_2, this path must include an arc $v_j v_i$ with $i < j$, which implies that $v_i v_{i+1} v_{i+2} \cdots v_j$ is a cycle of D. □

Hence, the only arc-traceable round digraph that is not strongly connected is a directed path.

Proposition 5.11. *A round digraph D that is not strongly connected is arc-traceable if and only if it is a directed path.*

To complete our brief analysis of arc-traceable round digraphs, we return to the case where $\delta^+ = 1$. We proceed in a similar way as we did for upset tournaments; we give necessary and sufficient conditions for a given arc to be traceable.

Theorem 5.12. *Let D be a strongly connected round digraph with round labeling $V = \{v_0, \ldots, v_{n-1}\}$. An arc $v_i v_j$ is traceable if and only if $j = i+1$, $d^+(v_{i-1}) \geq 2$ or $d^+(v_{j-1}) \geq 2$.*

Proof. Assume that $a = v_i v_j$ is traceable and $j \neq i+1$. Let H be a hamiltonian path of D containing this arc. Define the sets $V_1 = \{v_{i+1} v_{i+2} \cdots v_{j-2} v_{j-1}\}$ and $V_2 = \{v_{j+1} v_{j+2} \cdots v_{i-2} v_{i-1}\}$. If v_{i-1} precedes a on H, then H must include some arc $v_a v_b$ with $v_a \in V_2$ and $v_b \in V_1$. Thus, $v_{i-1} \in N^-(v_b)$ and so $d^+(v_{i-1}) \geq 2$. Similarly, if v_{i-1} follows the arc a on H, then there is some arc $v_a v_b$ on H with $v_a \in V_1$ and $v_b \in V_2$ and thus $v_{j-1} \in N^-(v_b)$ and $d^+(v_{j-1}) \geq 2$.

For the converse, assume first that $j = i + 1$. In this case, $v_i v_{i+1} \cdots v_{i-1}$ is a hamiltonian path containing a. Next, assume that $d^+(v_{i-1}) \geq 2$. Then,

$v_{i-1}v_{i+1}$ is an arc of D and $v_iv_jv_{j+1}\cdots v_{i-1}v_{i+1}v_{i+2}\cdots v_{j-1}$ is a hamiltonian path of D so long as $j \neq i-1$. In this case, $v_iv_jv_{i+1}v_{i+2}\cdots v_{i-2}$ is a hamiltonian path of D. Finally, if $d^+(v_{j-1}) \geq 2$, $v_{j-1}v_{j+1}$ is an arc of D and thus $v_{i+1}v_{i+2}\cdots v_{j-1}v_{j+1}\cdots v_{i-1}v_iv_j$ is a hamiltonian path of D for $j \neq i-1$ and $v_{i+1}v_{i+2}\cdots v_{i-2}v_iv_j$ is a hamiltonian path otherwise. □

We now take our characterization of traceable arcs and use it to obtain a characterization of arc-traceable strong round digraphs. Recall that $D_{(uv)}$ is the digraph obtained from D by removing the arc uv, if it is present.

Theorem 5.13. *Let D be a strong round digraph. Then D is arc-traceable if and only if $D - \{x, y\}$ is traceable and $D_{(yx)}$ has $\delta^- + \delta^+ > 0$ for every $xy \in A(D)$.*

Proof. Let $V = \{v_0, \ldots, v_{n-1}\}$ be a round labeling of D. Assume that D is arc-traceable and choose an arbitrary arc $v_iv_j \in A(D)$. Thus, by Theorem 5.12, $j = i+1$, $d^+(v_{i-1}) \geq 2$ or $d^+(v_{j-1}) \geq 2$.

First, we show that $D_{(v_jv_i)}$ has $\delta^- + \delta^+ > 0$. If $j \neq i-1$, then $D_{(v_jv_i)}$ is hamiltonian, and if $j = i-1$, then either $v_{j-1} \to v_i$ or $v_j \to v_{i+1}$ and so after removing the arc v_jv_i, we still have $d^+(v_j) > 0$ or $d^-(v_i) > 0$. Since the in-degree and out-degree of all other vertices remains unchanged and D is strong, it follows that $\delta^- + \delta^+ > 0$ in $D_{(v_jv_i)}$, as desired.

Next, we show that $D - \{x, y\}$ is traceable. If $j = i \pm 1$, then let P be the path $v_{i+2}v_{i+3}\cdots v_{n-1}v_0\cdots v_{i-2}$. Now, either Pv_{i-1} or $v_{i+1}P$ is a hamiltonian path of $D - \{x, y\}$. So we can assume that $j \neq i \pm 1$. If $d^+(v_{i-1}) \geq 2$, then $v_{i-1} \to v_{i+1}$ and $v_{j+1}\cdots v_{i-1}v_{i+1}\cdots v_{j-1}$ is a hamiltonian path of $D - \{x, y\}$. Similarly, if $d^+(v_{j-1}) \geq 2$, then $v_{i+1}\cdots v_{j-1}v_{j+1}\cdots v_{i-1}$ is a hamiltonian path of

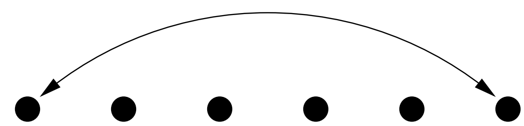


Figure 5.3: A strong non-arc-traceable round digraph.

$D - \{x, y\}$.

For the converse, assume that $D - \{v_i, v_j\}$ is traceable and $D_{(v_j v_i)}$ has $\delta^+ + \delta^- > 0$ for each $v_i v_j \in A(D)$. Let $V_1 = \{v_{i+1}, \ldots, v_{j-1}\}$ and $V_2 = \{v_{j+1}, \ldots, v_{i-1}\}$, and let H be a hamiltonian path of $D - \{v_i, v_j\}$. If neither V_1 nor V_2 are empty, then H must contain an arc $v_a v_b$ from V_1 to V_2 or from V_2 to V_1. Thus, either $v_{j-1} \in N^-(v_b)$ or $v_{i-1} \in N^-(v_b)$ and so $d^+(v_{j-1}) \geq 2$ or $d^+(v_{i-1}) \geq 2$. Hence, by Theorem 5.12 $v_i v_j$ is traceable. If V_1 is empty, then $j = i + 1$ and clearly, $v_i v_j$ is traceable. Finally, if V_2 is empty, then $j = i - 1$ and either $N^+(v_j) = \{v_i\}$ or $v_j \rightarrow v_{i+1}$. Additionally, either $N^-(v_i) = \{v_j\}$ or $v_{j-1} \rightarrow v_i$. As $\delta^+(D_{(v_j v_i)}) + \delta^-(D_{(v_j v_i)}) > 0$, we must have either $v_j \rightarrow v_{i+1}$ or $v_{j-1} \rightarrow v_i$. In the former case, $v_i v_j v_{i+1} \cdots v_{n-1} v_0 \cdots v_{i-2}$ is a hamiltonian path of D containing the arc $v_i v_j$. Similarly, in the latter case $v_{i+1} v_{i+2} \cdots v_{n-1} v_0 v_1 \cdots v_{j-1} v_i v_j$ is a hamiltonian path of D containing the required arc. Consequently, $v_i v_j$ is traceable and since this arc was arbitrary, D is arc-traceable. □

The need for the somewhat awkward "extra" condition on $D_{(yx)}$ in Theorem 5.13 is illustrated in Figure 5.3. The transitive tournament with an additional arc from x to y is strong, with a single non-traceable arc (the arc xy), but the digraph remaining when the vertices incident with this new arc are removed is certainly traceable. To see that this digraph is round, simply note that the transitive ordering of the original tournament is also a round labeling of the digraph.

5.3 Locally semicomplete digraphs

Turning our attention to locally semicomplete digraphs in general, we first consider the case of such digraphs that are not strongly connected. In this case, we use a result of Bang-Jensen.

Theorem N (Bang-Jensen [5]). *If D is a weakly connected (but not strongly connected) locally semicomplete digraph, then D^*, the condensation of D, is an acyclic round digraph and each strong component of D is a semicomplete digraph.*

From this we deduce a result about arc-traceability.

Corollary 5.14. *If D is a weakly connected (but not strongly connected) locally semicomplete digraph, then D is arc-traceable if and only if each strong component of D is an arc-traceable semicomplete digraph and D^*, the condensation of D, is a directed path.*

Once again, this is not quite a characterization, since it relies on a characterization of arc-traceable semicomplete digraphs, but we do know a great deal about the structure of non-arc-traceable strong semicomplete digraphs from Theorem 5.5.

In the case of strong locally semicomplete digraphs, we will focus on the sub-class known as *round decomposable* strong locally semicomplete digraphs. A round decomposition is somewhat more general than the condensation of a digraph. We will say a strong locally semicomplete digraph D has a *round decomposition* if there is a round digraph R with order r and the vertices of D can be partitioned into sets $S_1, \ldots, S_r$ in one-to-one correspondence with the vertices of R such that each S_i is a strong semicomplete digraph and that every vertex in S_i dominates each vertex in S_j whenever the vertex corresponding to S_i dominates the vertex corresponding to S_j.

We note that not all such digraphs have a round decomposition (see [16]). We will consider only those that do here. Round digraphs are certainly round decomposable, as we can simply choose $S_i = \{v_i\}$ for each vertex v_i. We use the techniques and arguments developed for round digraphs to give necessary and sufficient conditions for an arc in this larger class of digraphs to be traceable.

Lemma 5.15. *Let D be a strong round decomposable locally semicomplete digraph and let $S_1, \ldots S_r$ and R be a round decomposition of D. Then an arc $xy \in A(D)$ is non-traceable if and only if for some non-traceable arc $w_i w_j \in A(R)$, $S_i = \{x\}$ and $S_j = \{y\}$.*

Proof. Let D have the round decomposition specified in the theorem, and let $x \in S_i$ and $y \in S_j$.

If $i = j$, let H_i be a hamiltonian path of the semicomplete digraph induced on $S_i \setminus \{x, y\}$ and for $k \neq i$, let H_k be a hamiltonian path of the semicomplete digraph induced on S_k. Then $xyH_{i+1}H_{i+2} \cdots H_r H_1 H_2 \cdots H_i$ is a hamiltonian path of D that contains the arc xy.

If $i \neq j$ and $S_i \neq \{x\}$, then let H_i be a hamiltonian path of $S_i \setminus \{x\}$ and let H_j be a hamiltonian path of the strong semicomplete digraph induced on S_j beginning at y. Now, for $k \neq i, j$ let H_k be any hamiltonian path of S_k and construct the hamiltonian path $xH_jH_{j+1}\cdots H_rH_1H_2\cdots H_{j-1}$ of D containing the arc xy. Similarly, if $S_i = \{x\}$, but $S_j \neq \{y\}$, we use a similar technique and construct a hamiltonian path $H_{i+1}H_{i+2}\cdots H_nH_1\cdots H_{i-1}xy$.

Thus, we can assume that $S_i = \{x\}$ and $S_j = \{y\}$. It remains to show that in this case, xy is traceable in D if and only if w_iw_j is traceable in R. Clearly, if w_iw_j is traceable in R, the we can choose a hamiltonian path H_R containing this arc and replace each vertex w_k of this path with a hamiltonian path H_k of S_k and obtain a hamiltonian path of D that contains xy. Conversely, just as in the proof of Theorem 5.12, any hamiltonian path H of D containing xy must contain an arc uv with $u \in S_{k_1}$ and $v \in S_{k_2}$ where one of k_1, k_2 is in the set $\{j+1, j+2\ldots, i-1\}$ and the other is in the set $\{i+1, i+2, \ldots, j-1\}$. This forces the arc $w_{i-1}w_{i+1}$ or the arc $w_{j-1}w_{j+1}$ in R and this arc is sufficient to guarantee that w_iw_j is traceable in R by Theorem 5.12. □

This gives us precisely the same characterization of arc-traceable strong round decomposable locally semicomplete digraphs that we obtained in the case of round digraphs.

Theorem 5.16. *Let D be a strong round decomposable locally semicomplete digraph. Then D is arc-traceable if and only if $D - \{x, y\}$ is traceable and $D_{(yx)}$ has $\delta^- + \delta^+ > 0$ for every $xy \in A(D)$.*

Proof. The result follows from Theorem 5.15 using the exact same argument used in the proof in Theorem 5.13. □

To complete an analysis of all arc-traceable locally semicomplete digraphs, such digraphs that are not round decomposable must be considered. This case remains open.

Question. *Characterize arc-traceable locally semicomplete digraphs that are neither semicomplete nor round decomposable.*

5.4 Other Generalizations

In this last section we pose a series of questions that are related to arc-traceability in tournaments. We will suggest other classes of digraphs in which arc-traceability could be investigated, and we will consider properties similar to arc-traceability. For clarity, we will state each problem in the context of all digraphs. Of course, many such questions could be asked. We present only a small sample.

Perhaps the most obvious generalization of tournaments not already discussed is the class of *multipartite tournaments.* A multipartite tournament is an orientation of a complete multipartite graph.

Question. *Characterize arc-traceable multipartite tournaments.*

Another possibility is to look at directed graphs which fall just short of tournaments. For example, digraphs with at most one pair of non-adjacent vertices.

Question. *Characterize arc-traceable digraphs whose underlying graph is* $K_n - e$.

Generalizing the idea of arc-traceability itself, we arrive at a fundamental question: Given any substructure S, what digraphs D have a hamiltonian path containing S for each S in D? We briefly consider a few choices for the structure S.

Question. *Characterize digraphs D that contain a hamiltonian path containing S for each S in D where S is:*

- *A path of length $k, k \geq 2$.*
- *k vertex disjoint arcs, in any order.*
- *An ordered set of k vertex disjoint arcs.*
- *A linear forest with possibly additional restrictions.*

We can also generalize the "traceable" part of arc-traceable. Instead of requiring a single, spanning path, we can seek digraphs with a linear factorization containing the desired structure. Such a generalization may be necessary to make significant progress on these stronger conditions, as there remain many open questions even in the case of arc-traceable digraphs. One might also consider factorizations containing the given structure that consist of both paths and cycles. Path-cycle factorizations of semicomplete multipartite digraphs have been studied by Bang-Jensen, Huang, Gutin and Yeo (for an overview of these results, we suggest [6] and for specific references, [7] and [17]).

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