University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

August 27, 2005

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{C}$ denotes the set of complex numbers. These extend to vector spaces as $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{C}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	__________	5.	__________
2.	__________	6.	__________
3.	__________	7.	__________
4.	__________	8.	__________
		Total	__________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Lynn Bennethum, Weldon Lodwick, Jan Mandel (Chair)

1. Let (A, d) be a metric space, $K \subset A$ a nonempty compact set, and $x \in A \setminus K$. Show that there exists $y \in K$ such that $d(x, y)$ is minimal.

2. Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and $x \in D$. Prove that f is continuous at x (using the $\varepsilon - \delta$ definition) if and only if for every sequence $\{x_n\}$, $x_n \in D$ $\forall n$, $x_n \to x$ implies $f(x_n) \to f(x)$.

3. Consider the function defined by $f(x) = \sin \frac{1}{x}$, $x > 0$, $f(0) = 0$. Using the definition of Riemann integral, show that f is Riemann integrable on the interval $[0, 1]$.

4. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

5. Let $\{a_n\}$ be a monotonically decreasing sequence of real numbers $a_n > 0$ such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $na_n \to 0$, $n \to \infty$.

6. Consider the sequence of functions $f_n(x) = n^2 x e^{-nx}$, where $n = 1, 2, ...$, and $x \in [0, 1]$.

   (a) Determine the pointwise limit, $f(x) = \lim_{n\to\infty} f_n(x)$.

   (b) Calculate $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx$.

   (c) Calculate $\int_0^1 \lim_{n\to\infty} f_n(x)\, dx$.

   (d) Explain your results, proving statements where appropriate.

7. Show that a compact metric space has a countable dense subset.

8. (Implicit Function Theorem in 1D) Consider $F(x,y) : \mathbb{R}^2 \to \mathbb{R}$ where $F \in C^1([x_0 - \delta, x_0 + \delta] \times [y_0 - \delta, y_0 + \delta])$. Assume that $F(x_0, y_0) = 0$ and that $F_y(x_0, y_0) = \dfrac{\partial F}{\partial y}(x_0, y_0) > 0$. Prove that there exists a unique function $f(x)$ such that $F(x, f(x)) = 0$ in a neighborhood about (x_0, y_0).